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SPACE-CHARGE LIMITING CURRENTS IN A DRIFT TUBE OF FINITE LENGTH--ETC(U)
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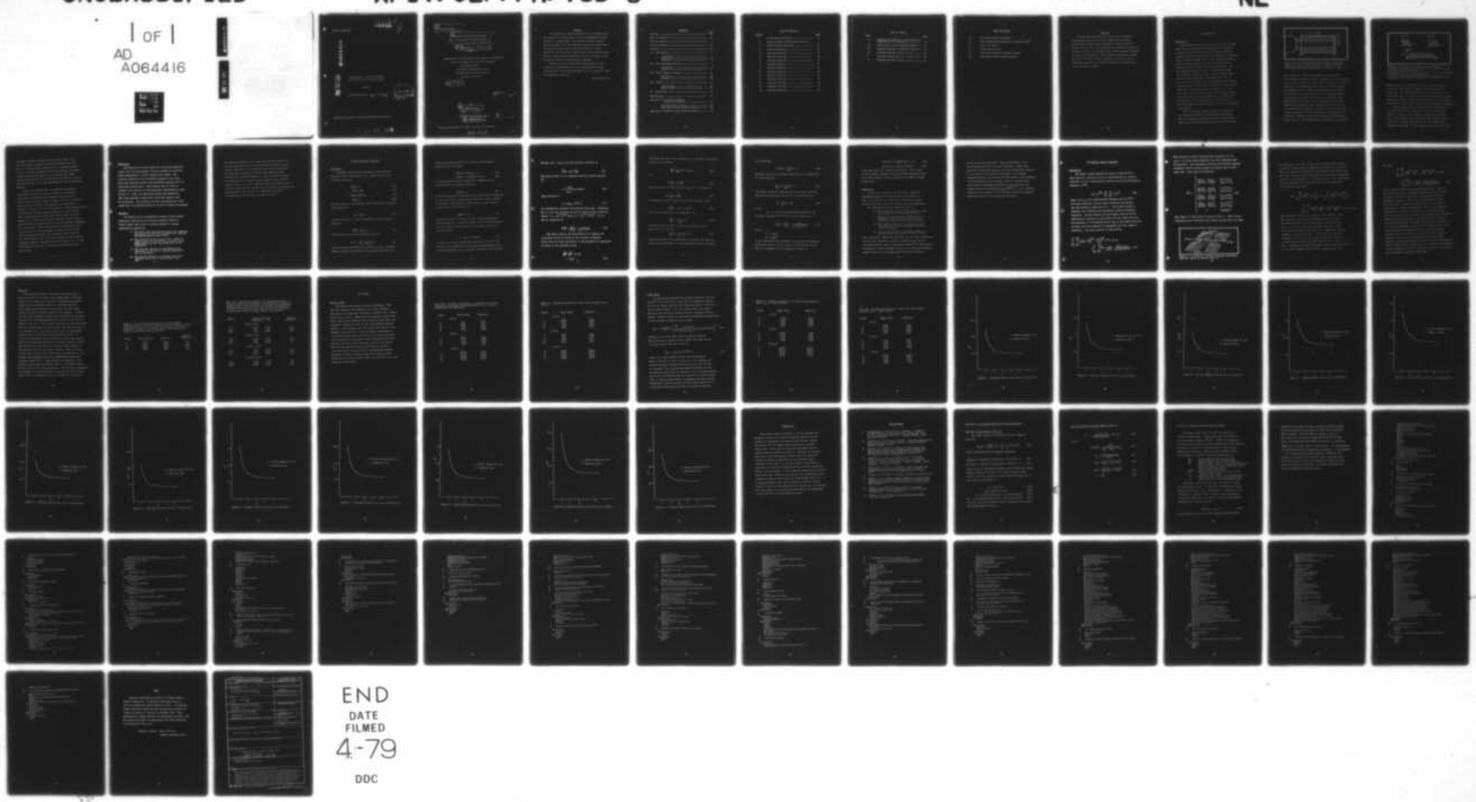

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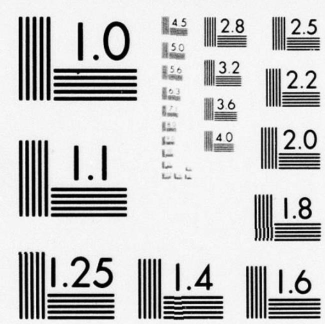

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6 Space-Charge Limiting Currents
in a Drift Tube of Finite Length.

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THESIS

Presented to the Faculty of the School of Engineering
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in Partial Fulfillment of the
Requirements for the Degree of
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Preface

At present very little information is available about the effects of finite length on the space-charge limiting current in a drift tube. This report provides accurate numerical values for this limiting current for drift tubes of several lengths and for electron beams of several geometries and energies. These values may be used to check various analytical approximations for the limiting current and in the design of experimental equipment.

Thanks are due to Professor Philip Nielsen, Thomas Genoni, and William Proctor for their guidance on this project and to Professor John Jones for his assistance with the numerical technique.

Steve Harrington

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List of Symbols

ϕ	the electrostatic potential
φ	the electrostatic potential in units of e/mc^2
e	1.6×10^{-19} coul.
I	total beam current
ν	total beam current in units of e/mc^3
ν_L	the space-charge limiting current

Abstract

The equation governing the steady state potential distribution created by a cold unneutralized annular relativistic electron beam propagating axially within a finite length drift tube with grounded anode foil, collector plate, and walls and immersed in an axially directed infinite magnetic guide field is solved numerically using the method of finite elements. These numerical results are used to check the accuracy of available analytical approximations to the limiting current.

I Introduction

Background

Many applications of intense relativistic electron beams (x-ray and microwave generation, thermonuclear fusion experiments, and collective ion acceleration schemes, among others) require that the beam be injected into an evacuated drift tube. If an external magnetic guide field is applied, the injected electrons may propagate thru the drift tube as a stable beam. If the external magnetic field is sufficiently strong (as defined in Section II) the maximum current which can be totally transmitted thru the drift tube in steady state is limited by the electron number density (space-charge) at the center of the tube. For currents above the space-charge limiting current, the space-charge at the center of the tube becomes large enough to reflect part of the electron beam. In this paper the effects of the finite length of the drift tube on the space-charge limiting current will be considered numerically.

For a drift tube (Fig. 1) with $L \gg R$ and with B_o effectivly infinite, Bogdankevich and Rukhadze (Ref. 1) formulated an analytical approximation for the space-charge limiting current of a solid electron

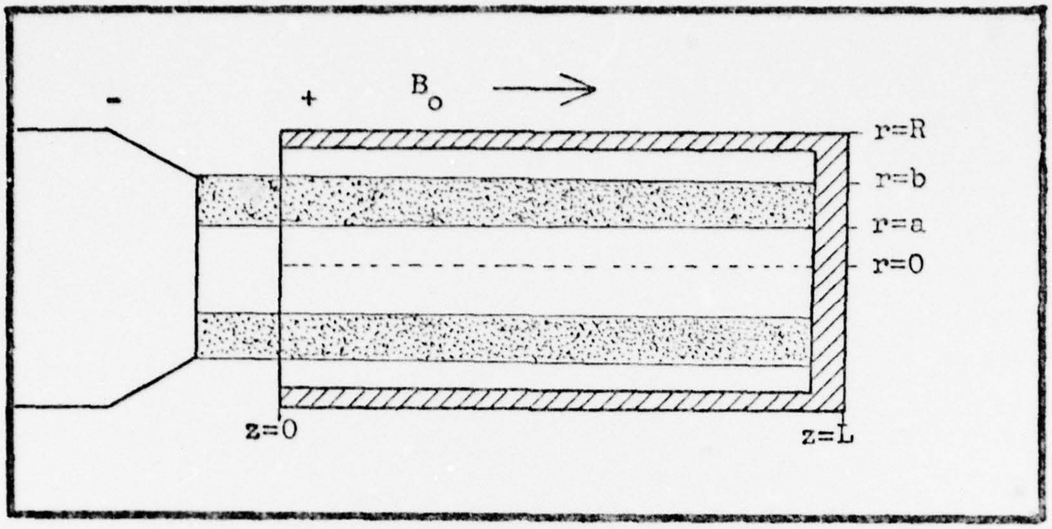

Figure 1. An electron drift tube of finite length (L) in an effectively infinite axially directed magnetic guide field (B_o). The electrons are injected at z=0 and occupy the drift tube between r=a and r=b. (Ref. 2)

beam (Fig. 2a). Their approximation was derived by interpolating between the exact solutions for nonrelativistic and ultrarelativistic electron beams. However, both exact solutions were derived assuming that the number density distribution shown in Fig. 2a propagated unchanged thru the drift tube. These solutions were derived subject to the further assumption that L>>R and, therefore, the potential is constant in the z direction near the tube center as shown in Fig. 2c. Proctor and Genoni (Ref. 3) later derived an analytical approximation to the limiting current analogous to the one derived by Bogdankevich and Rukhadze. However, in their derivation annular electron beams (Fig 2b) were considered in addition to solid beams.

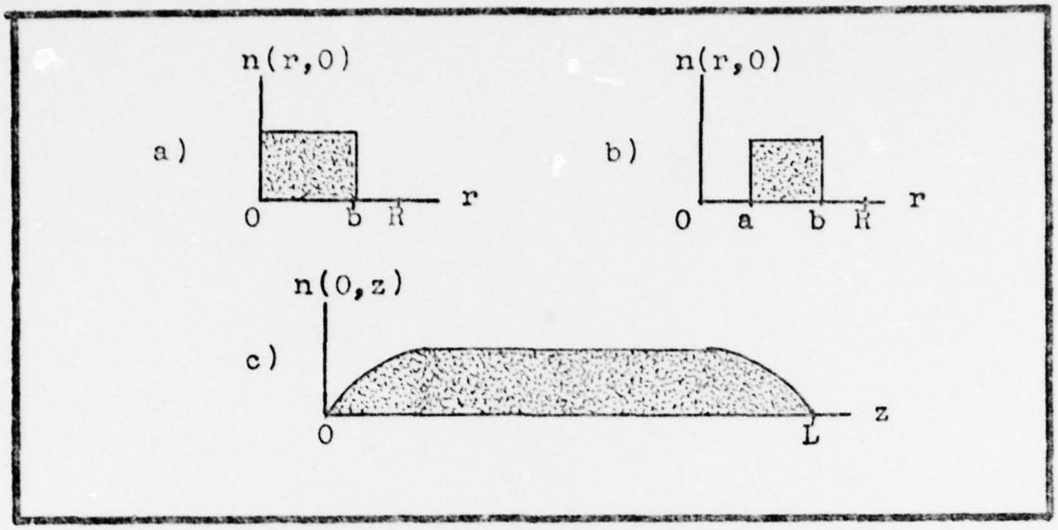

Figure 2. Electron number density distributions.
a) radial number density distribution of a solid electron beam at the injection anode.
b) radial number density distribution of an annular (hollow) electron beam at the injection anode.
c) axial number density distribution of an electron beam propagating in a long (L 5R) drift tube (Ref. 2).

Furthermore, their derivation does not require that the electron number density propagate thru the drift tube unchanged. The expression for the limiting current in a long drift tube (as derived by Proctor and Genoni) is given in Appendix A and was used to check the numerical solutions of the long length limit of the equation derived in Section II.

Voronin et al. (Ref. 4) relaxed the restriction that the drift tube be long compared to the tube radius and derived a rigorous upper bound to the limiting current of a solid electron beam filling the drift space (Fig 2a with b=R). Proctor and Genoni (Ref. 5) extended this analysis to include a length dependent upper bound to the

limiting current of annular electron beams (Fig. 2b). In both of these analyses the current density ($n\underline{v}$) and not the electron number density (n) was assumed constant in z. It should be emphasized that the Proctor and Genoni formula given in Appendix A is a rigorous upper bound but not necessarily a least upper bound to the space-charge limiting current. It is not an expression for the limiting current.

Proctor and Genoni (Ref. 6) derived an analytical approximation for the limiting current of an electron beam with an electron number density proportional to a delta function in the radial direction (Fig. 2b with $a \rightarrow b$). This is still, however, essentially a one dimensional problem. Miller (Ref. 7) has developed a fully two dimensional approximation for the limiting current of solid electron beams propagating in finite length drift tubes (given in Appendix A). In this derivation, however, Miller assumed that the electron number density remained constant throughout the drift tube. Further, he defined the limiting current as that current which produced a potential drop (due to the space-charge) between the injection anode (z=0) and the tube center (z=L/2) equal in magnitude to the electron injection energy. However, it has been shown (Ref. 2) that the potential drop at the limiting current need not be that great.

Objectives

The objectives in this paper are to present numerical values for the space-charge limiting currents of electron beams propagating in finite length drift tubes. The limiting currents of two different geometry solid beams with each geometry having three different energies will be determined numerically. These values will be used to estimate the accuracy of the analytical approximation given in Ref. 7. Also, two different hollow beam geometries with each geometry having three different energies will be considered. The limiting currents determined for these cases will be presented mostly as an aid to future development

Approach

In Section II the differential equation with boundary conditions describing the electric potential inside a finite length drift tube is derived subject to several simplifying assumptions:

1) The drift tube and electron source are immersed in a large (as defined in Section II), axially directed magnetic guide field.

2) The electron kinetic energy ($(\gamma - 1)mc^2$) is large enough (as described in Section II) to make scattering from the anode foil at z=0 negligible.

3) The electron velocity is the same for all electrons at z=0 and is totally directed in the +z direction.

4) The current density is constant across the electron beam (r=a to r=b in Fig. 1) at z=0.

The numerical analog to the equation derived in Section II is derived in Section III and the numerical method (finite element method) used to solve this equation is discussed. The limiting currents for several beam geometries and energies are given in Section IV. These limiting currents are compared to the rigorous upper bound for the limiting currents given in Ref. 5. For the solid beams the limiting currents determined numerically are compared to the analytical approximation given in Ref. 7. Finally, in Section V recommendations for further research are made.

II The Governing Equation

Derivation

Maxwell's steady state equations for the electromagnetic fields in the drift tube shown in Fig. 1 (in gaussian units) are

$$\nabla \times \underline{E} = 0 \tag{1a}$$

$$\nabla \cdot \underline{E} = 4\pi en \tag{1b}$$

$$\nabla \times \underline{B} = \frac{4\pi}{c} ne\underline{v} \tag{1c}$$

$$\nabla \cdot \underline{B} = 0 \tag{1d}$$

where n and $\underline{v}$ are the electron number density and velocity. In steady state

$$\underline{E} = -\nabla \phi \tag{2}$$

Equations (1) and (2) can be combined to give Poisson's equation

$$\nabla^2 \phi = 4\pi enU(r) \tag{3}$$

where U(r) is a unit step function such that

$$U(r) = \begin{cases} 0 & a > r,\ b < r \\ 1 & a \leq r \leq b \end{cases} \tag{4}$$

By electrically connecting the injection anode, collector plate, and drift tube walls and by holding then all at a

constant potential (defined to be zero), the boundary conditions on Eq (3) are

$$\phi(r,0) = \phi(r,L) = 0 \tag{5a}$$

$$\phi(R,z) = 0,\ \phi(0,z)\ \text{finite} \tag{5b}$$

The divergence of Eq (1c) yields the equation for the steady state conservation of charge:

$$\nabla\cdot(n\underline{v}) = 0 \tag{6}$$

If the electrons are injected with axially directed velocity, are not scattered significantly by the anode foil, and the drift tube is immersed in a sufficiently strong magnetic guide field, Eq (6) reduces to

$$\frac{\partial}{\partial z}(nv) = 0 \tag{7}$$

which means that the charge density (nv) is independent of axial position. Therefore, since nv is radially constant at the injection anode (z=0), it is constant throughout the drift tube:

$$nv = n_o v_o = \text{constant} \tag{8}$$

By multiplying and dividing the right hand side (RHS) of Eq (3) by the velocity of an electron and applying Eq (8), Poisson's equation can be expressed in terms of the current

density ($env = en_o v_o$) and the electron velocity as

$$\nabla^2 \phi = 4\pi e \frac{n_o v_o}{v} \qquad (9)$$

The beam current (I) is computed from the current density by

$$I = \int_a^b \int_0^{2\pi} (env)(r dr d\theta) \qquad (10)$$

which reduces to

$$I = en_o v_o \, (b^2 - a^2) \qquad (11)$$

for cylindrical geometry and constant beam area. Combining Eq (11) and the velocity written in terms of the relativity factor ($v = c\sqrt{1 - \gamma^{-2}}$ where $\gamma = 1/\sqrt{1 - (v/c)^2}$) Eq (9) can be expressed as

$$\nabla^2 \phi = \frac{4I}{b^2 - a^2} \; \frac{1}{c\sqrt{1 - \gamma^{-2}}} \qquad (12)$$

The final step in the derivation is to express the relativity factor in terms of the electric potential. To do this the total derivitive of the potential is expressed in terms of the electric field:

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla \phi \\ &= -\underline{v} \cdot \underline{E} \end{aligned} \qquad (13)$$

By taking the total time derivative of the total relativistic energy of an electron

$$\frac{d\mathcal{E}^2}{dt} = \frac{d}{dt}(c^2P^2 + m^2c^4) \tag{14}$$

or

$$\mathcal{E}\,\frac{d\mathcal{E}}{dt} = c^2\underline{P}\,\frac{dP}{dt} \tag{15}$$

By expressing $\mathcal{E}$ as γmc^2 and $\underline{P}$ as $\gamma m\underline{v}$, Eq (15) reduces to

$$mc^2\,\frac{d\gamma}{dt} = \underline{v}\cdot\frac{d\underline{P}}{dt} \tag{16}$$

By forming the dot product of $\underline{v}$ with the Lorentz force

$$\underline{v}\cdot\frac{dP}{dt} = -e\underline{v}\cdot(\underline{E} + \frac{v}{c}\times\underline{B}) \tag{17}$$

Eq (16) becomes (noting that $\underline{v}\cdot(\underline{v}\times\underline{B}) = 0$)

$$\frac{d\gamma}{dt} = -\frac{e}{mc^2}\,\underline{v}\cdot\underline{E} \tag{18}$$

Equations (13) and (18) can now be combined to form a constant of the motion:

$$\frac{d}{dt}\left(\gamma - \frac{e}{mc^2}\phi\right) = 0 \tag{19}$$

By expanding the total derivative in Eq. (19) into the sum of partial derivatives, and by noting that this is a steady

state problem,

$$\underline{v} \cdot \nabla(\gamma - \frac{e}{mc^2}\phi) = 0 \tag{20}$$

However, since $\underline{v}$ is totally directed in the +z direction, Eq (20) reduces to

$$\frac{\partial}{\partial z}(\gamma - \frac{e}{mc^2}\phi) = 0 \tag{21}$$

Therefore, since the potential at z=0 is zero, and since the term in parenthesis in Eq (21) is constant in z,

$$\gamma - \frac{e}{mc^2}\phi = \gamma_o \tag{22}$$

where

γ_o is the electron normalized energy at z=0

Finally, using Eq (22), Eq (12) can be expressed as

$$\nabla^2\varphi = \frac{4\nu}{b^2-a^2} \frac{U(r)}{\sqrt{1-(\gamma_o+\varphi)^{-2}}} \tag{23}$$

where

$\nu = eI/mc^3$

$\varphi = e\phi/mc^2$

It is numerically more convenient to express the boundary conditions from Eq (5) in terms of the potential at z=0 and the derivative of the potential at z=L/2 as

$$\varphi(r,0) = 0, \ \frac{d}{dz}\varphi(r,L/2) = 0 \qquad (24a)$$

$$\varphi(R,z) = 0, \ \varphi(R,0) \text{ finite} \qquad (24b)$$

which are valid for a symetric potential. This change in the boundary conditions permit greater accuracy in the numerical solution without increasing the number of mesh points used.

Discussion

It has been shown that not all values of ν permit physically meaningful solutions to Eq (23) (Ref. 5). That is, solutions for which the potential is nowhere positive and zero only at the boundary of the drift tube.

The assumptions used to derive Eq (23) were

1) The externally imposed, axial magnetic guide field (B_0) may be considered infinite.

2) The kinetic energy of the electrons at the injection anode is large enough to make scattering from the anode foil negligible.

3) The electron velocity is the same for all electrons at z=0 and is totally directed in the +z direction.

4) The current density is constant across the electron beam (r=a to r=b in Fig. 1) at z=0.

From a practical standpoint, the first two of these assumptions are the most difficult to obtain. Thode et al. (Ref. 8) has quantified the terms "considered infinite" and "negligible scattering." Their findings are that, for a typical injection anode scattering can be neglected if the total particle

energy is greater than $2mc^2$. This is equivalent to an accelerating field of 0.5 MV for electrons. Also, they show that the magnetic guide field may be considered infinite if the ratio of the cyclctron frequence to the plasma frequency (w_c/w_p, a measure of the gyroradius of the particle) at the injection anode is greater than 5. In this ratio the cyclotron frequency is defined by $w_c = eB_o/mc$ and the plasma frequency is defined by $w_p = \sqrt{4\pi ne^2/m}$. For a typical electron number density ($10^{12}/cm^3$) this requires a magnetic guide field greater than about 75 kilo-gauss.

III Finite Element Solution

Derivation

The finite element method was used to solve Eq (23). This numerical technique involves approximating the unknown solution (φ) to a differential equation with an interpolating function (φ^{NM}):

$$\varphi \approx \varphi^{NM} = \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij} S^{ij} \qquad (25)$$

where the c_{ij}'s are undetermined constants and the S^{ij}'s are known functions (called basis functions) satisfying the same boundary conditions as φ. The method chosen to determine the constants was Galerkin's method of weighted residuals. By this method the approximate solution to the differential equation is substituted for the exact solution, the equation is then multiplied by each of the basis functions in turn, and each equation is integrated over the range of variables. One such equation is shown below:

$$\int_{r=0}^{R} \int_{z=0}^{L/2} \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \varphi^{NM}}{\partial r} + \frac{\partial^2 \varphi^{NM}}{\partial z^2} \right) S^{ij} \, r \, dr \, dz$$

$$= \int_{r=0}^{R} \int_{z=0}^{L/2} U(r) \frac{4V}{b^2 - a^2} \frac{S^{ij} \, r \, dr \, dz}{\sqrt{1 - (\gamma_0 + \varphi^{NM})^{-2}}} \qquad (26)$$

This produces N times M simultaneous equations for the c_{ij}'s. To solve these equations the basis functions must be specified. For the present problem the bilinear Lagrange polynomial basis functions given by Prenter (Ref. 9:128) were used. This basis is defined by

$$
S^{ij} = \begin{cases}
\dfrac{r_{i+1}-r}{r_{i+1}-r_i} \quad \dfrac{z-z_{j-1}}{z_j-z_{j-1}} & \begin{array}{l} r_i \le r \le r_{i+1} \\ z_{j-1} \le z \le z_j \end{array} \\[2ex]
\dfrac{r_{i+1}-r}{r_{i+1}-r_i} \quad \dfrac{z_{j+1}-z}{z_{j+1}-z_j} & \begin{array}{l} r_i \le r \le r_{i+1} \\ z_j \le z \le z_{j+1} \end{array} \\[2ex]
\dfrac{r-r_{i-1}}{r_i-r_{i-1}} \quad \dfrac{z-z_{j-1}}{z_j-z_{j-1}} & \begin{array}{l} r_{i-1} \le r \le r_i \\ z_{j-1} \le z \le z_j \end{array} \\[2ex]
\dfrac{r-r_{i-1}}{r_i-r_{i-1}} \quad \dfrac{z_{j+1}-z}{z_{j+1}-z_j} & \begin{array}{l} r_{i-1} \le r \le r_i \\ z_j \le z \le z_{j+1} \end{array} \\[2ex]
0 & \text{elsewhere}
\end{cases} \tag{27}
$$

One element of this basis is shown in Fig. 3. These linear polynomial basis functions were chosen because they are among

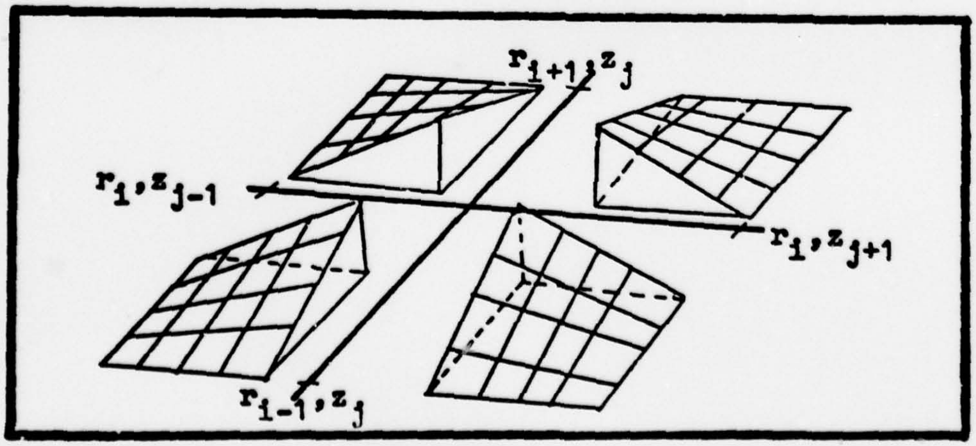

Figure 3. A canonical bilinear Lagrange polynomial over its region of support (Ref.9).

the simplest to evaluate numerically and because they produce a strongly banded system of algebraic equations to be solved. That is a system of equations in which most of the coefficients are zero. However, using this basis does introduce a difficulty with the second derivative - it doesn't exist. This difficulty can be eliminated by integrating the left hand side (LHS) of Eq (26) once by parts:

$$\text{LHS} = \int_{z=0}^{L/2} \left(r \frac{\partial \varphi^{NM}}{\partial r} S^{ij} \Big|_{r=0}^{R} \right) dz + \int_{r=0}^{R} \left(r \frac{\partial \varphi^{NM}}{\partial z} S^{ij} \Big|_{z=0}^{L/2} \right) dr$$

$$- \int_{r=0}^{R} \int_{z=0}^{L/2} \left(\frac{\partial \varphi^{NM}}{\partial r} \frac{\partial S^{ij}}{\partial r} + \frac{\partial \varphi^{NM}}{\partial z} \frac{\partial S^{ij}}{\partial z} \right) r \, dr \, dz \qquad (28)$$

The integrated portions of Eq (28) can be evaluated using the boundary conditions for the differential equation. The first integral is obviously zero at r equal to zero. For r equal to R, the basis functions must all be zero to match the boundary conditions on φ. Therefore, the first integral is identically zero. The second integral also evaluates to zero since the basis functions are zero at z equal to zero and the derivative is zero at z equal to L/2. Therefore, the only contribution from the LHS of Eq (26) is the third integral from Eq (28). Combining Eqs (26)

and (28),

$$-\int_{r=0}^{R}\int_{z=0}^{L/2}\left(\frac{\partial\varphi^{NM}}{\partial r}\frac{\partial S^{ij}}{\partial r}+\frac{\partial\varphi^{NM}}{\partial z}\frac{\partial S^{ij}}{\partial z}\right) r\,dr\,dz$$

$$=\int_{r=0}^{R}\int_{z=0}^{L/2} U(r)\,\frac{4\nu}{b^2-a^2}\,\frac{S^{ij}\;r\,dr\,dz}{\sqrt{1-(\gamma_0+\varphi^{NM})^{-2}}} \qquad (29)$$

is the actual equation to be solved numerically. Since i and j vary over their respective ranges of values, Eq (29) actually represents a system of simultaneous non-linear algebraic equations. This set of equations was solved by a non-linear equation solving program (QN) written by programmers at Sandia Laboratories. Each of the equations represented by Eq (29) was integrated numerically using assumed values for the c_{ij}'s and a two dimensional trapezoidal integration approximation. These equations for the c_{ij}'s were solved by QN using a quasi-Newton iterative technique in which the equations were repeatedly linearized and solved until successive iterates converged. The space-charge limiting current was defined numerically as the largest current for which the iterative equation solver could produce a converging solution that was physically meangingful: that is, for which the potential was less than zero everywhere in the interior of the drift tube and for which the potential met the boundary conditions specified.

Accuracy

The numerical equation (Eq (29)) was solved using a mesh size of 14 by 7 in the r and z directions. The mesh points were equally spaced in the z direction and equally spaced in the r direction with the exception that the mesh points falling closest to the inner and outer beam radius was forced to fall exactly at that radius. To check to see if the numerical solution was converging to one value, the mesh size was increased up to the limit of the computer memory (15 by 15). These results were in very close agreement with the results using a 14 by 7 mesh. The limiting currents were computed to a precision of ± 0.01 units of normalized current (ν) or about 170 amps of electron current. In Table I the limiting currents found by solving this equation for the numerical approximation to a delta function electron distribution are compared to the results given in Ref. 6. The maximum error is less than 2 percent. Table II shows the limiting currents for a drift tube of length 5R. At this length, the limiting current has essentially reached its infinite length value. These results are compared to the infinite length results given in Ref. 2. The maximum error in this case is less than 3 percent. Based on these comparisons the maximum error in the numerical results for finite length drift tubes is estimated to be on the order of 5 percent.

Table I. The limiting currents for a delta function distribution of electrons is compared to the limiting currents found numerically for an electron beam having a geometry of a=0.4999, b=0.5001. Both beams are centered about r=0.5 and have an injection energy of $20mc^2$.

Length	Delta Function	Numerical	Percent Difference
1	7.30	7.43	1.7
2	4.60	4.62	0.4
3	4.07	4.06	0.2
5	3.81	3.81	0.0
∞	3.78	3.75	0.8

Table II. The limiting currents for electron beams of several energies and geometries propagating in an infinite length drift tube are compared to the limiting currents for electron beams propagating in a drift tube of length 5R. A length of 5R was used since that was the longest length in which any effects of finite length were observed.

Energy	Limiting Current (infinite)	Limiting Current 5R	Percent Difference
	a=0.0, b=0.5		
2.0	0.22	0.22	0.0
5.0	1.27	1.29	1.5
20.0	7.31	7.37	0.8
	a=0.0, b=1.0		
2.0	0.55	0.56	1.8
5.0	3.12	3.16	1.3
20.0	17.58	17.75	1.0
	a=0.57, b=0.60		
2.0	0.43	0.44	2.3
5.0	2.53	2.56	1.1
20.0	15.19	15.50	2.0
	a=0.30, b=0.60		
2.0	0.32	0.33	3.0
5.0	1.87	1.91	2.0
20.0	10.91	11.12	1.9

IV Results

Annular Beams

Two annular beam geometries were considered. They were the numerical approximation to a delta function distribution of electrons ($a=0.4999$, $b=0.5001$) and a thicker beam ($a=0.4$, $b=0.6$). Both beams were centered about $r=0.5$. Tables III (delta function) and IV (thicker beam) give the results of solving Eq (29) for the space-charge limiting current. Also presented are the upper bounds given by the formula derived in Ref. 5 and reproduced in Appendix A. It should be noted that in all cases the limiting current does approach the numerically correct infinite drift tube limit as the tube length increases and that the results for the thin annulus do agree with previous results. At present there is no analytical approximating formula available for the limiting current in an annular beam. Therefore, the data in these tables are presented as an aid to future researchers.

Table III. Limiting currents for a geometry of a=0.4999, b=0.5001 (a numerical approximation to a delta function distribution of electrons).

Length	Upper Bound	Numerical
$\gamma_0 = 2.0$		
1	0.71	0.66
2	0.44	0.41
3	0.38	0.35
5	0.34	0.33
∞	0.33	0.32
$\gamma_0 = 8.0$		
1	8.13	7.43
2	5.02	4.62
3	4.33	4.06
5	3.96	3.81
∞	3.75	3.75
$\gamma_0 = 20.0$		
1	25.16	22.61
2	15.52	14.10
3	13.39	12.43
5	12.25	11.76
∞	11.59	11.59

Table IV. Limiting currents for thick annular beams ($a=0.4$, $b=0.6$)

Length	Upper Bound	Numerical
$\gamma_0 = 2.0$		
1	0.86	0.80
2	0.50	0.46
3	0.42	0.39
5	0.38	0.36
∞	0.36	0.35
$\gamma_0 = 8.0$		
1	9.93	8.79
2	5.72	5.09
3	4.86	4.41
5	4.41	4.13
∞	4.16	4.02
$\gamma_0 = 20.0$		
1	30.70	26.43
2	17.69	15.32
3	15.04	13.31
5	13.65	12.53
∞	12.85	12.28

Solid Beams

Two solid beam geometries were also considered (filling and half filling the drift tube) and the results of solving Eq (29) for these geometries are given in Tables V (filling) and VI (half filling). As with annular beams, the upper bounds to the limiting currents are also presented in these tables. However, Miller has derived an analytical approximation to the limiting currents of solid beams:

$$\nu_L(L) = \frac{\beta(\gamma_o - 1)}{8}\, b \left\{ \sum_{n=1}^{\infty} \frac{J_1(\lambda_n b)}{\lambda_n^3 \left[J_1(\lambda_n) \right]^2} \left[1 - \operatorname{sech}(\lambda_n L/2) \right] \right\}^{-1} \qquad (30)$$

where λ_n is the n'th zero of the J_o Bessel function.
This equation is approximated by Miller for tubes having a length greater than the radius by

$$\nu_L(L) = \nu_L (1 - 2e^{-\lambda_1 L/2})^{-1} \qquad (31)$$

where ν_L is the infinite length limiting current.
Figures 4-9 show Eq (30) as compared to the numerically derived limiting currents and Figures 10-15 show Eq (31) as compared to the numerically derived limiting currents. It should be noted that the infinite length limiting current in Eq (31) was obtained from the Proctor and Genoni formula (Ref. 3) and not from Miller's expression for that current. Comparison of these two sets of graphs shows that Eq (31) is in much better agreement with the numerical results.

Table V. Limiting currents for a solid beam filling the drift tube. (a=0.0, b=1.0).

Length	Upper Bound	Numerical
	$\gamma_0=2.0$	
1	1.76	1.42
2	0.93	0.69
3	0.75	0.60
5	0.70	0.57
∞	0.65	0.57
	$\gamma_0=5.0$	
1	10.44	7.72
2	5.51	3.76
3	4.59	3.26
5	4.12	3.10
∞	3.86	3.08
	$\gamma_0=20.0$	
1	63.88	42.90
2	33.15	20.89
3	27.64	18.10
5	24.82	17.21
∞	23.23	17.12

Table VI. Limiting currents for a solid beam half filling the drift tube. (a=0.0, b=0.5)

Length		Upper Bound	Numerical
	$\gamma_0 = 2.0$		
1		0.58	0.53
2		0.32	0.26
3		0.27	0.23
5		0.25	0.21
∞		0.23	0.21
	$\gamma_0 = 5.0$		
1		3.47	3.10
2		1.92	1.51
3		1.62	1.31
5		1.47	1.24
∞		1.38	1.24
	$\gamma_0 = 20.0$		
1		20.70	17.80
2		11.55	8.67
3		9.76	7.51
5		8.83	7.14
∞		8.30	7.10

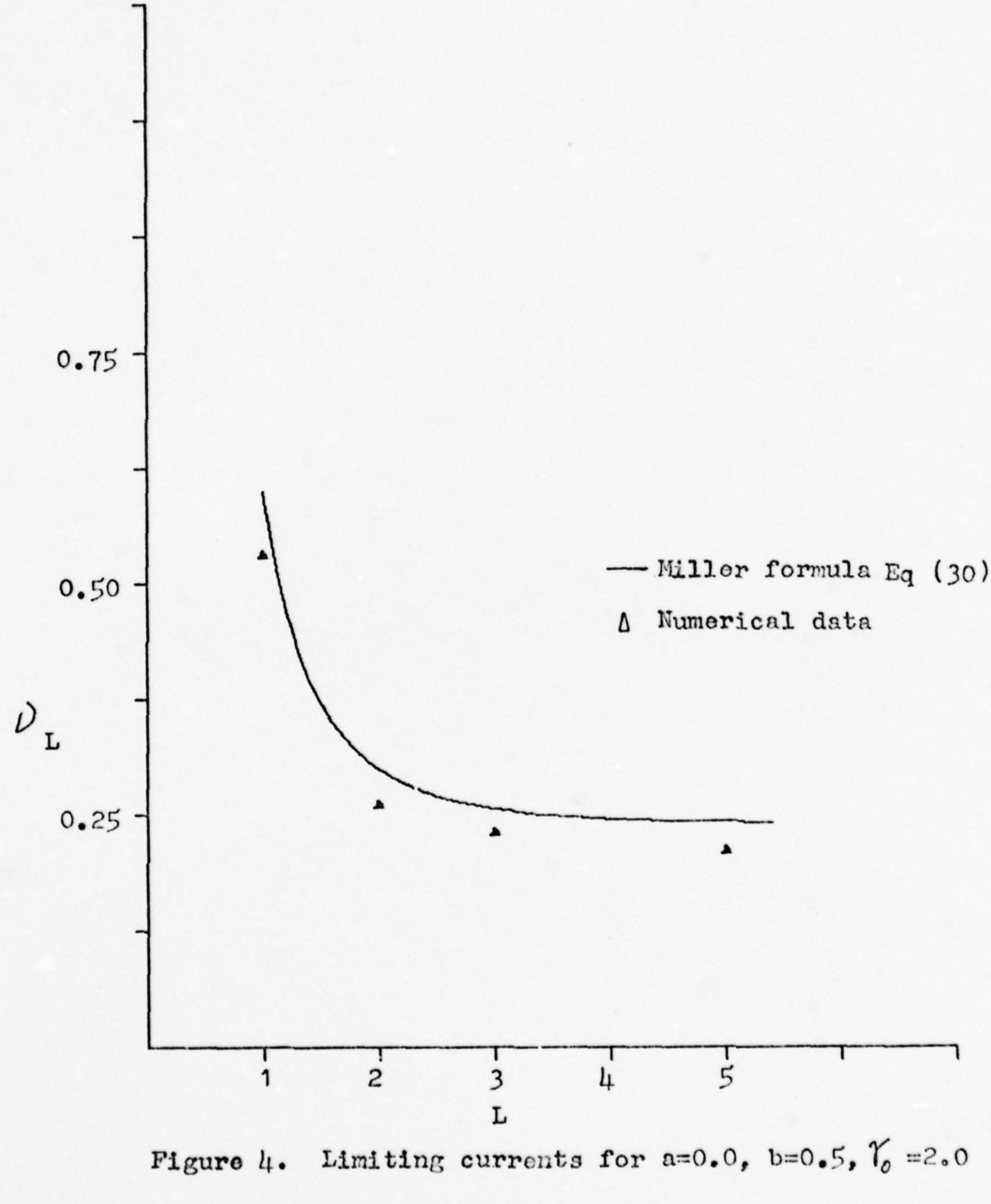

Figure 4. Limiting currents for a=0.0, b=0.5, γ_0 =2.0

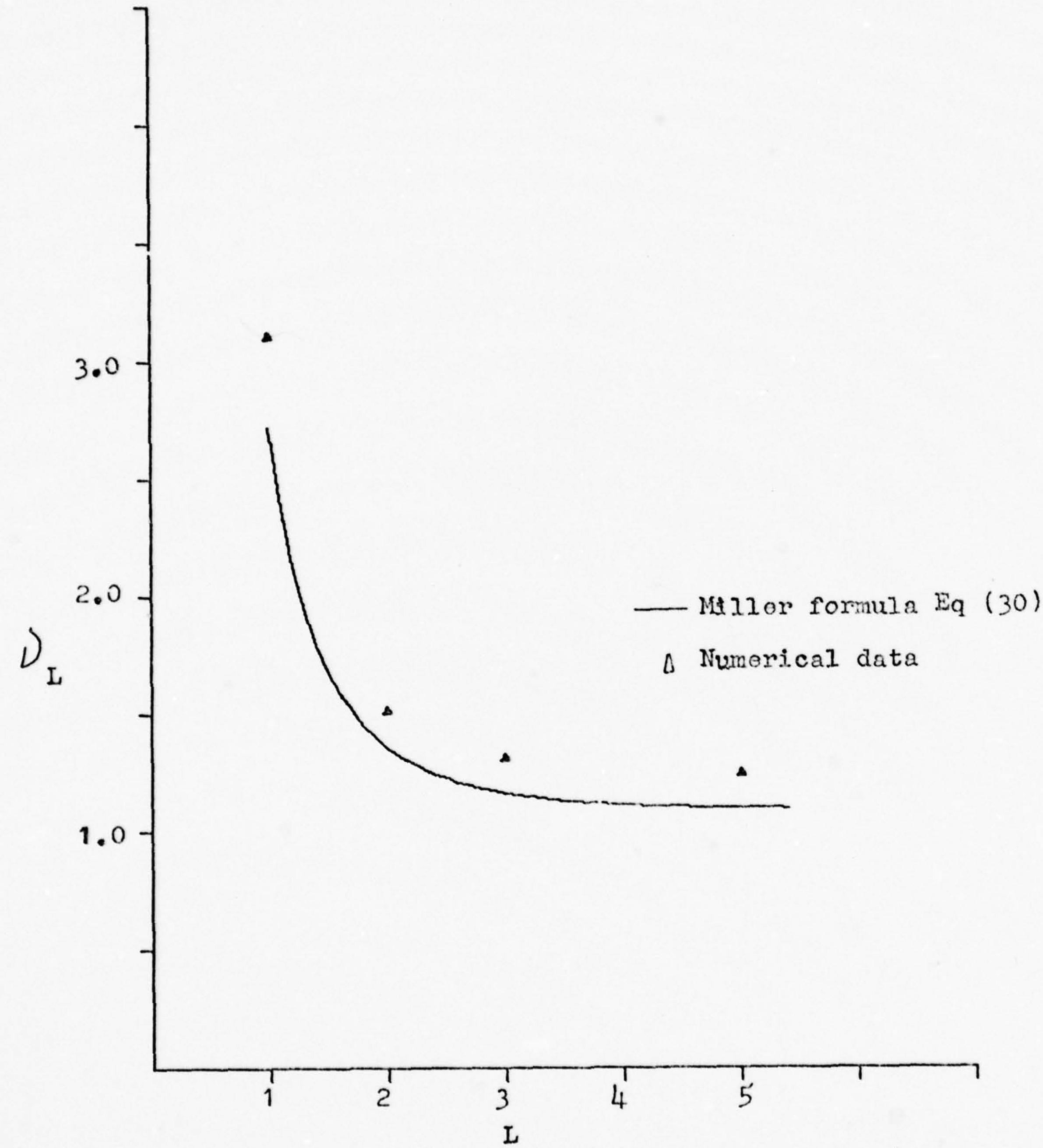

Figure 5. Limiting currents for a=0.0, b=0.5, γ_0=5.0

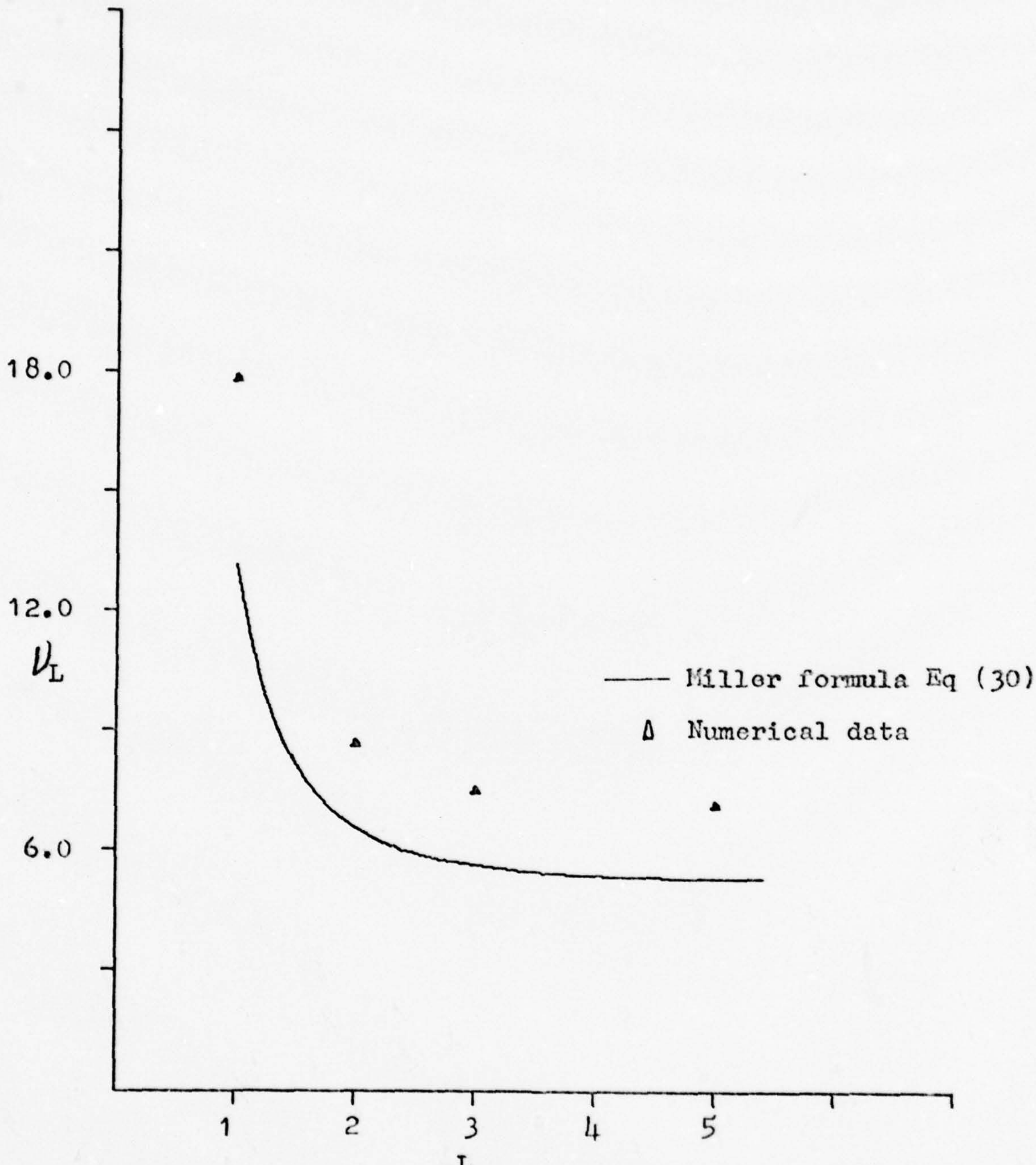

Figure 6. Limiting currents for a=0.0, b=0.5, γ_o=20.0

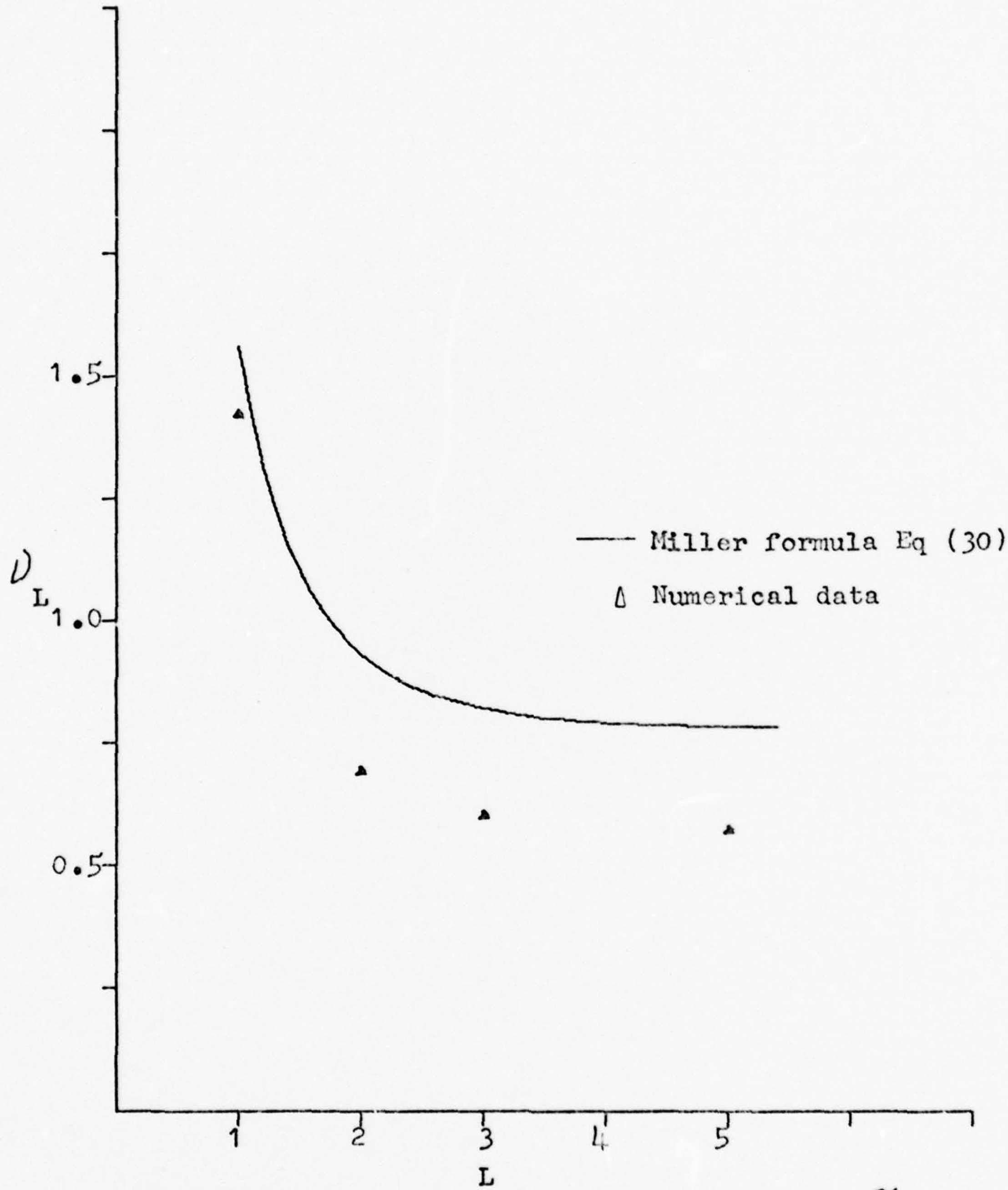

Figure 7. Limiting currents for a=0.0, b=1.0, γ_0=2.0

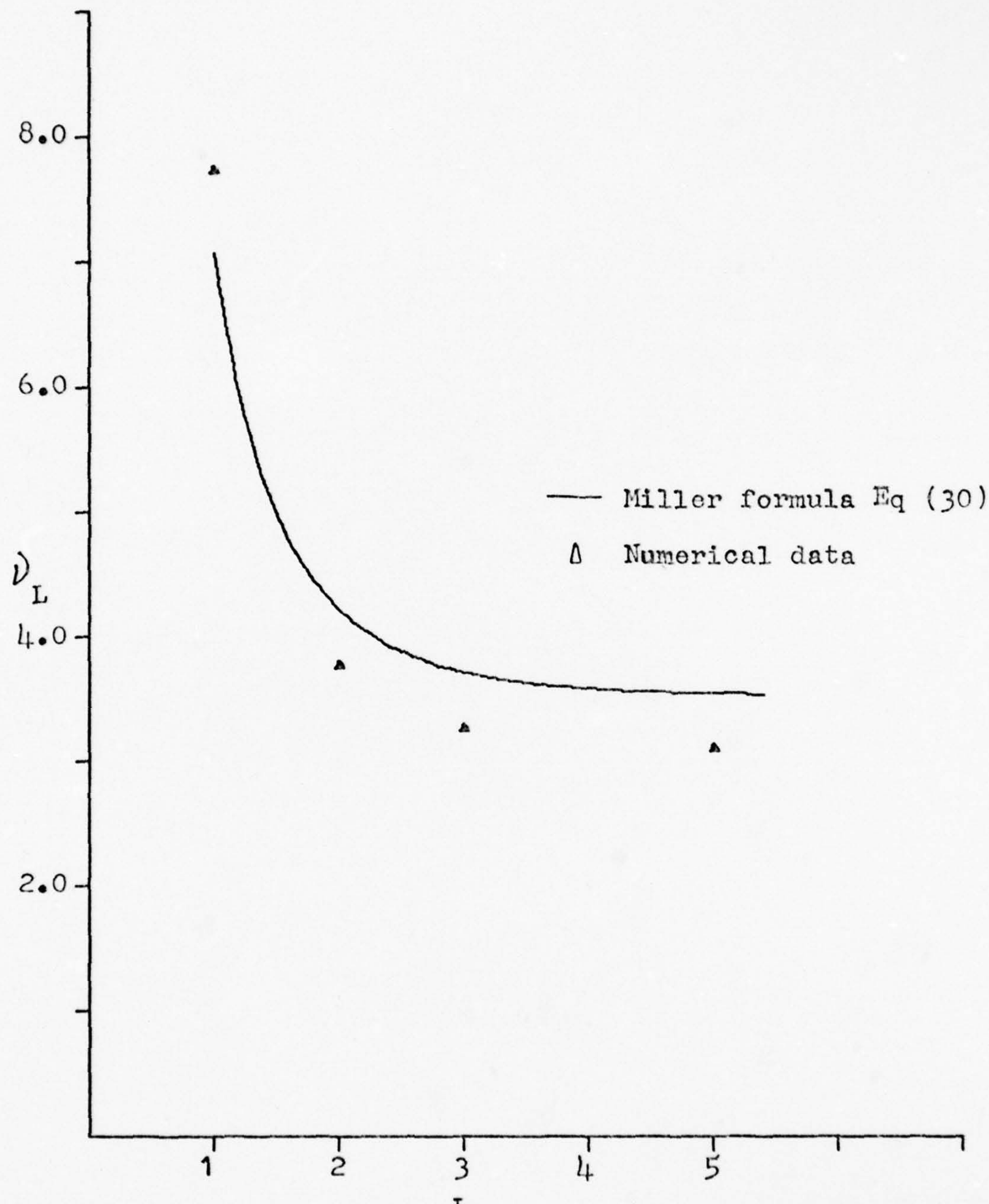

Figure 8. Limiting currents for a=0.0, b=1.0, γ_0=5.0

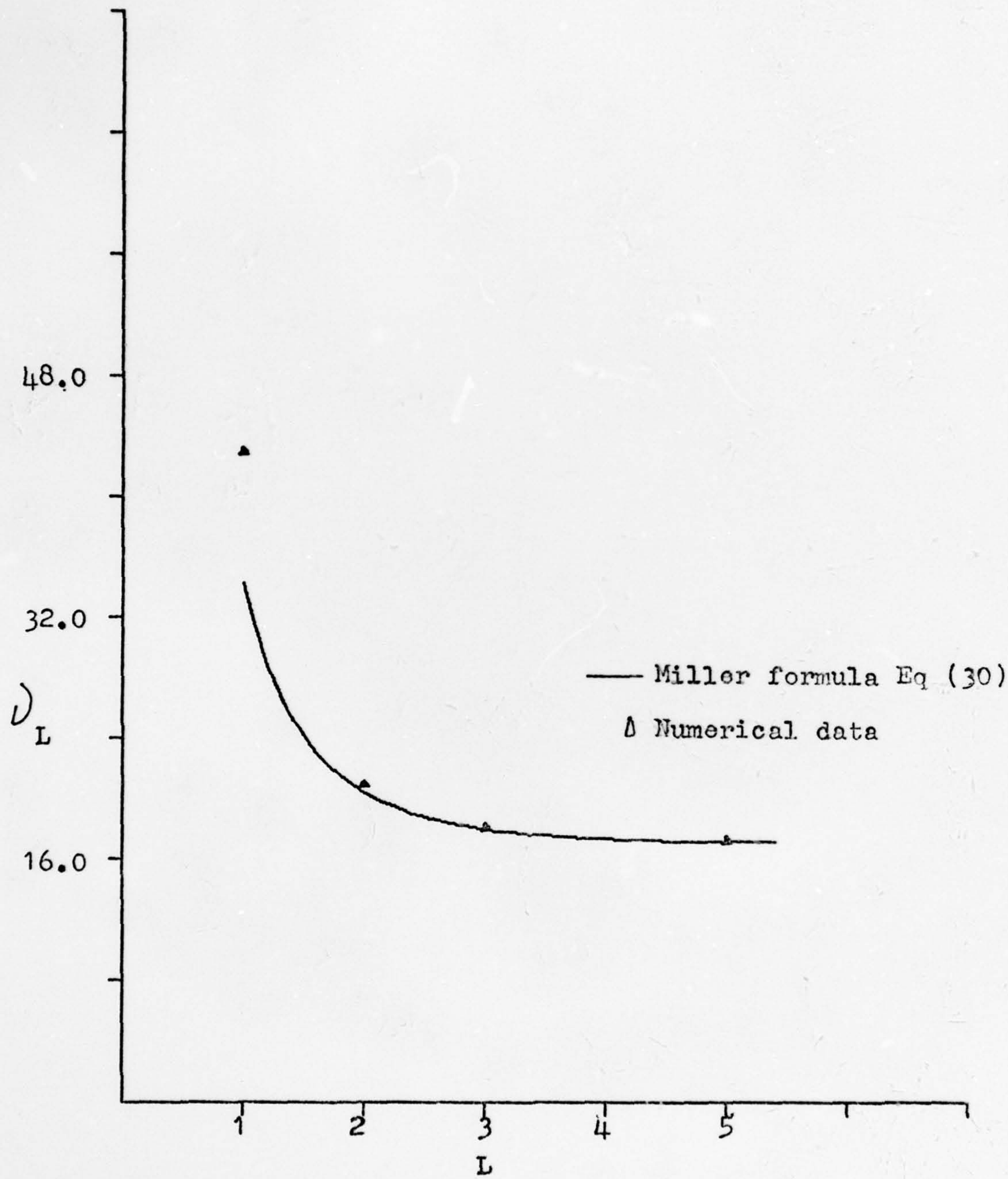

Figure 9, Limiting currents for a=0.0, b=1.0, γ_0=20.0

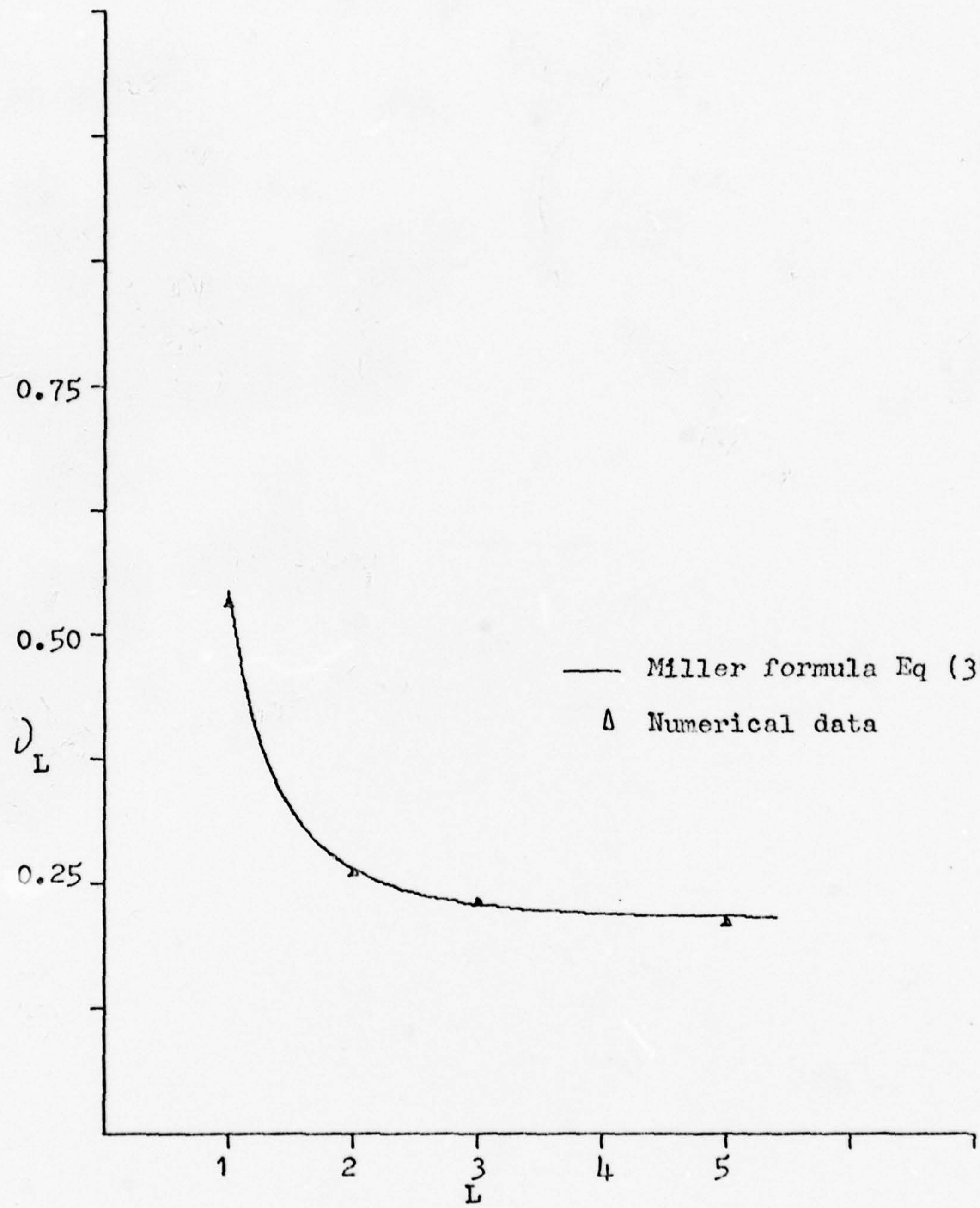

Figure 10. Limiting currents for a=0.0, b=0.5, γ_0=2.0

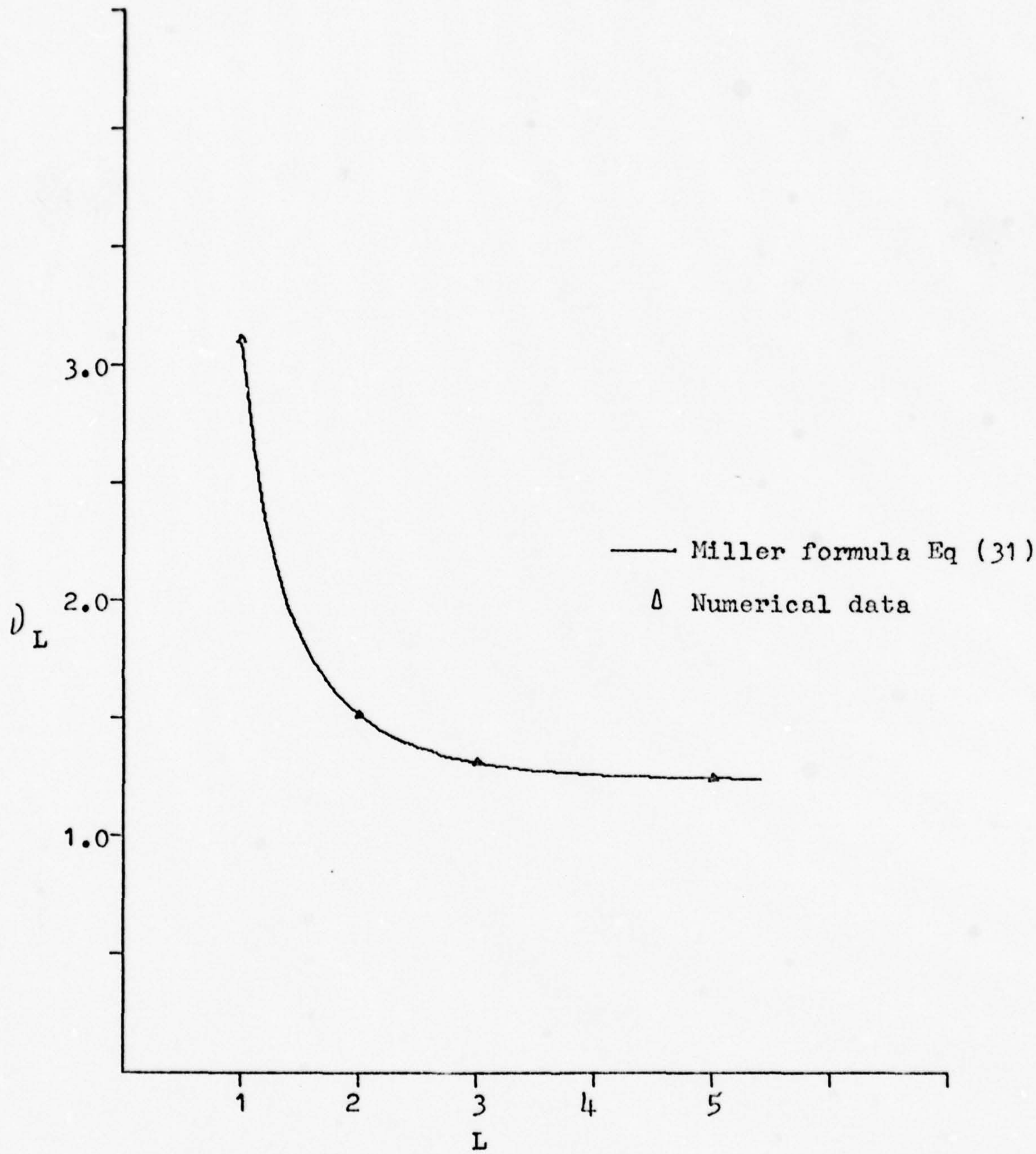

Figure 11. Limiting currents for a=0.0, b=0.5, γ_0=5.0

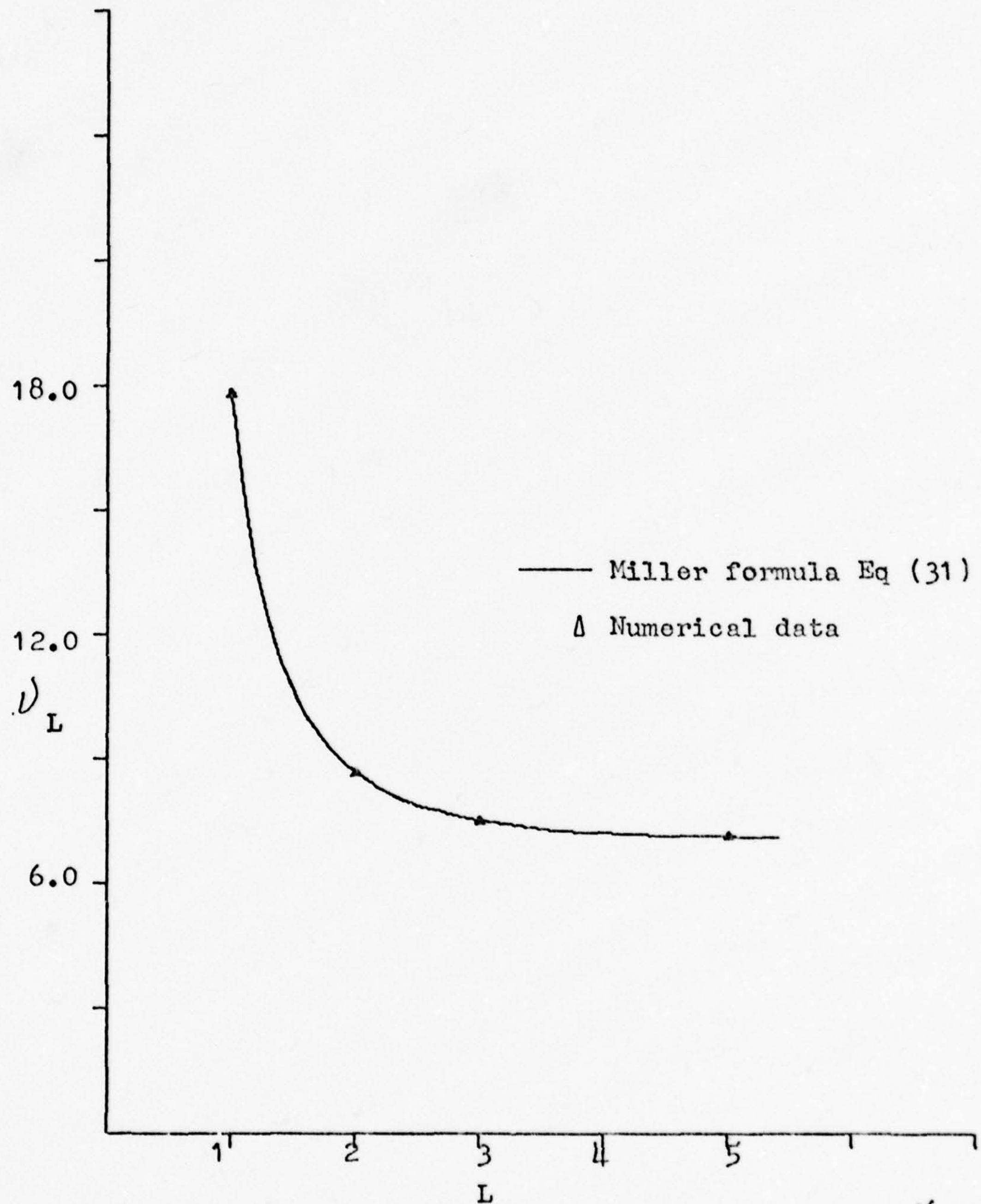

Figure 12. Limiting currents for a=0.0, b=0.5, γ_0=20.0

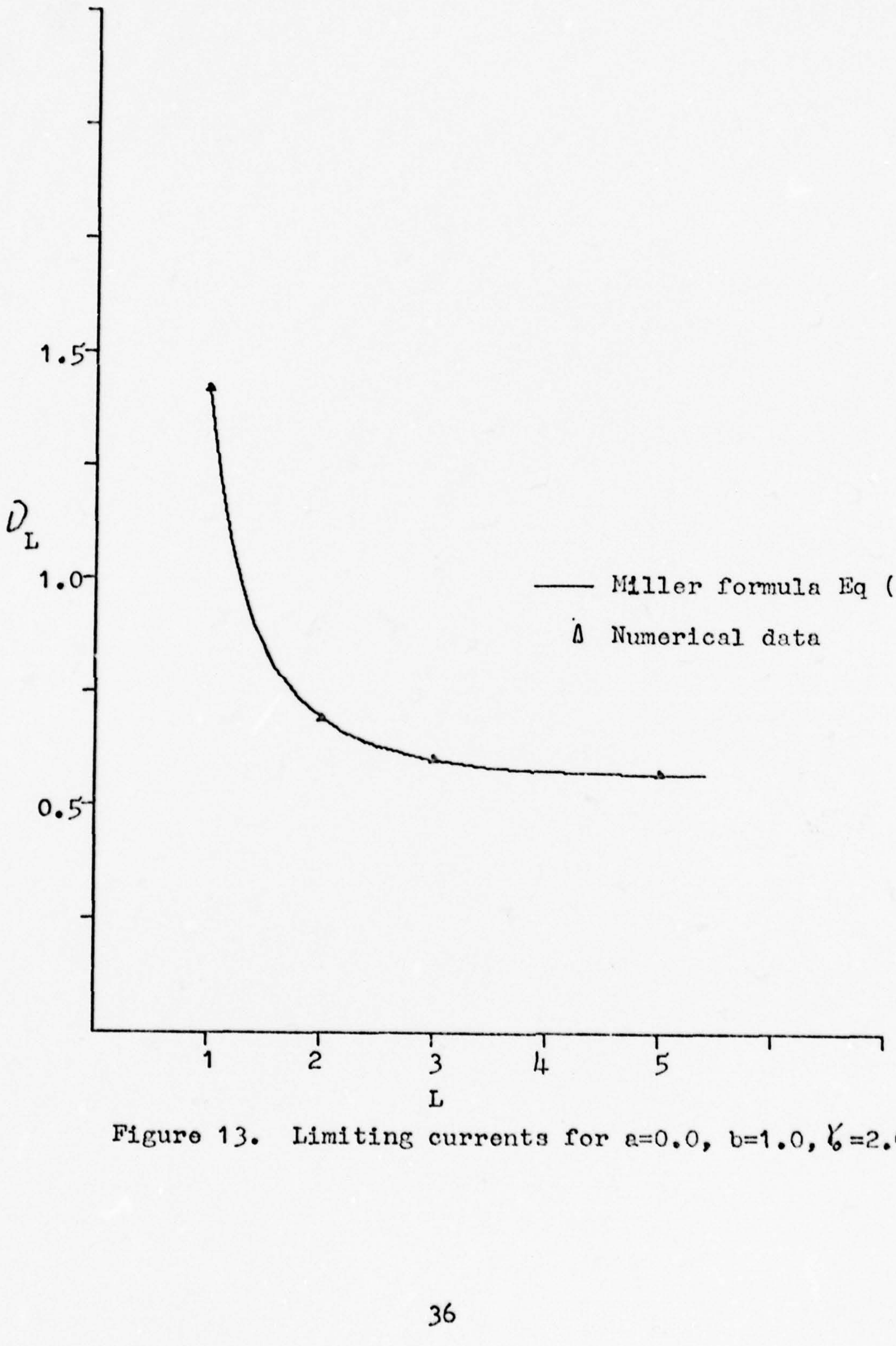

Figure 13. Limiting currents for a=0.0, b=1.0, γ_0=2.0

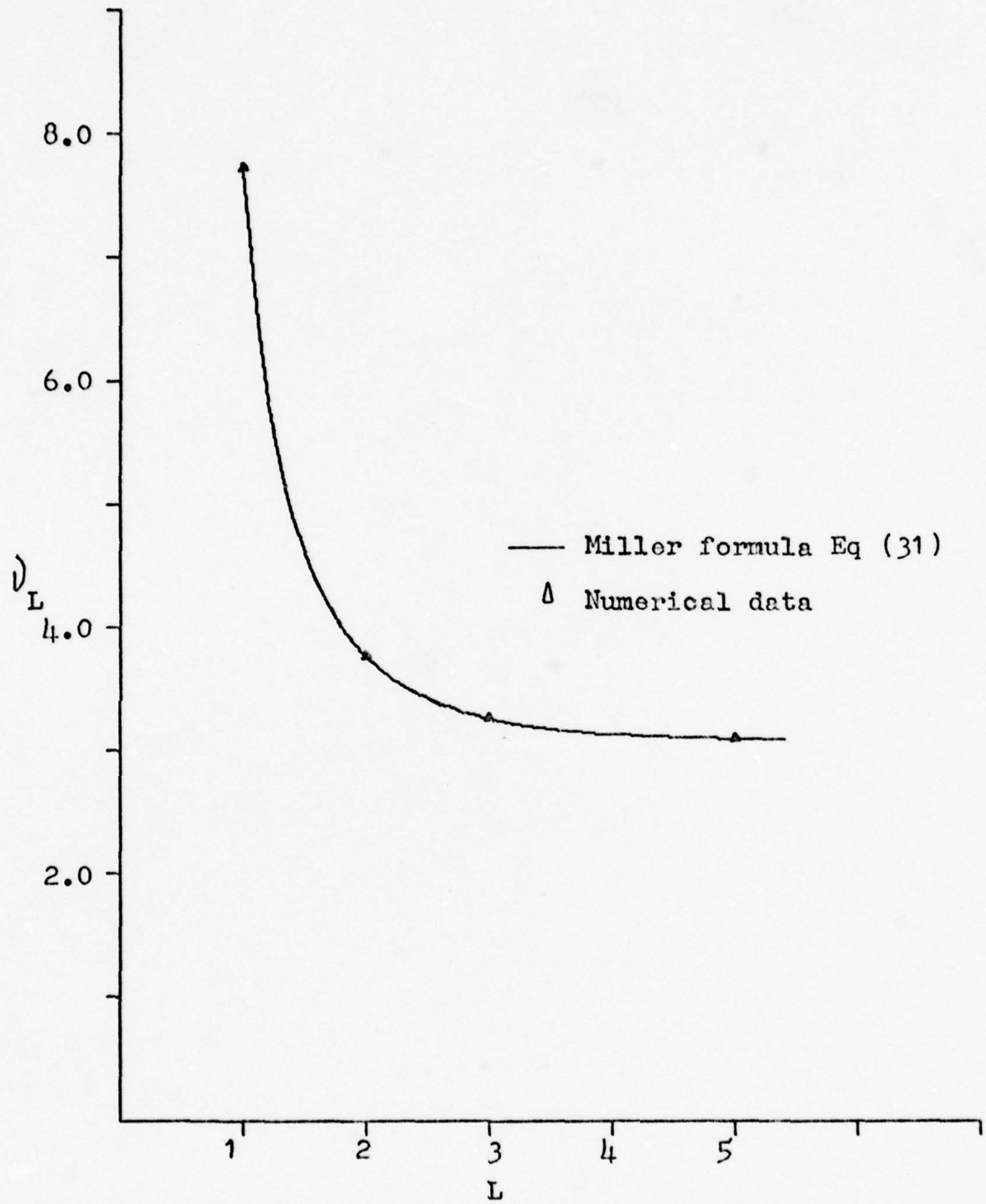

Figure 14. Limiting currents for a=0.0, b=1.0, γ=5.0

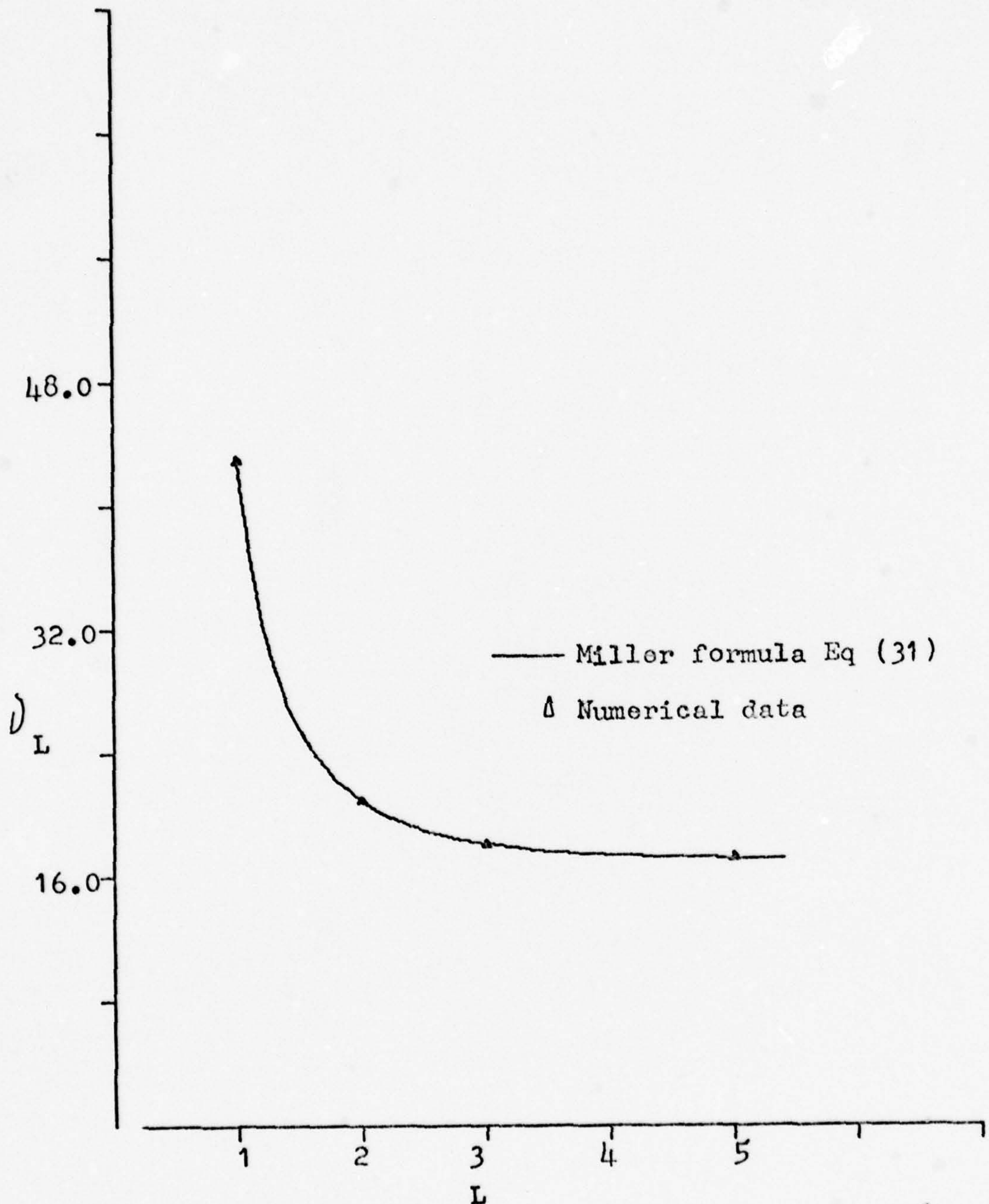

Figure 15. Limiting currents for a=0.0, b=1.0, γ_0=20.0

V Conclusion

The results shown in Section IV and the program that generated then form a complete numerical solution to the problem of determining the space-charge limiting current in a drift tube of finite length immersed in a strong guide field. These results show good agreement with the formula derived by Miller in Ref. 7. However, by comparing the results from the simpler formula (Eq (31)) using the Proctor and Genoni expression for the infinite length limiting current (Ref. 3) with the results using Miller's expression as derived it appears obvious that the assumptions Miller used (n constant, at the limiting current $|\varphi(0,L/2)| = \gamma_0 - 1$) and Proctor and Genoni did not use are less accurate then the assumptions used in this paper (nv constant, $|\varphi(0,L/2)| \leq \gamma_0 - 1$) It seems reasonable that any extension of Miller's formula should be based on the assumption that n<u>v</u> and not n is constant in z and should use a different definition of the conditions on the potential at the limiting current.

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Appendix A: Analytical Formula from the Literature

The Upper Bound Formula (Ref. 5)

The upper bound to the limiting current (ν_{UB}) is given by

$$\nu_{UB}(L) = \frac{b^2-a^2}{4} (k^2+ \frac{\pi^2}{L^2})(\gamma_o^{2/3}-1)^{3/2} \qquad (A1)$$

with k determined by the boundary conditions

$$a_2\left[J_o(ka) + CY_o(ka)\right] + a_1 ka\left[J_1(ka) + CY_1(ka)\right] = 0 \qquad (A2a)$$

$$b_2\left[J_o(kb) + CY_o(kb)\right] + b_1 kb\left[J_1(kb) + CY_1(kb)\right] = 0 \qquad (A2b)$$

where J_m and Y_m are Bessel functions of the first and second kind of order m, a and b are the inner and outer beam radius, and C is a constant to be determined along with k. The a_i and b_i are given by

$$a_1 = I_o(\pi a/L) \qquad (A3a)$$

$$a_2 = (\pi a/L)I_1(\pi a/L) \qquad (A3b)$$

$$b_1 = K_o(\pi/L)I_o(\pi b/L) - I_o(\pi/L)K_o(\pi b/L) \qquad (A3c)$$

$$b_2 = (\pi b/L)\left[K_o(\pi/L)I_1(\pi b/L) + I_o(\pi/L)K_1(\pi b/L)\right] \qquad (A3d)$$

where I_m and K_n are modified Bessel functions of the first and second kind of order m.

The Space-Charge Limiting Current (Ref. 3)

$$\nu_l = \frac{(\gamma_0^{2/3}-1)^{3/2}}{1-f(E)+2\ln(1/b)} \, G(\gamma_0, \Delta) \tag{A4}$$

where

$$G(\gamma_0, \Delta) = \frac{\gamma_0^{2/3}}{\left[(\gamma_0^{2/3}+\Delta)^2\right]^{1/2}-\Delta} \tag{A5}$$

$$\Delta = \frac{1+2\ln(1/b)-f(E)}{1-g(E)} \tag{A6}$$

$$f(E) = \frac{(1-E)^2}{1-E/2} \left| \frac{\ln(1-E)}{E} \right| \tag{A7}$$

$$g(E) = \frac{8(1-E)^2}{E(4-3E)} \, \frac{\ln(1-E/2)}{1-E} \tag{A8}$$

$$E = \frac{b-a}{b} \tag{A9}$$

Appendix B: Finite Element Computer Program

The program that solves Eq (29) was written as a series of subprograms. These subprograms are called by the supervising program (DRIVER). DRIVER is responsible for reading all parameter cards, insuring that the parameters are valid, computing the locations of the mesh points, calling the appropriate equation solver (ONED or TWOD), and printing the results. The valid parameters are

RO	inner radius of the electron beam
RN	outer radius of the electron beam
NR	number of mesh points in the r direction
V	beam current. This is an optional parameter which, if omitted, causes DRIVER to solve for the limiting current
GAMO	electron injection energy
ZN	length of the drift tube (required for two dimensional runs)
NZ	number of mesh points in the z direction (also required for two dimensional runs)
ID	1 or 2 for one or two dimensional runs

Subroutine TWOD (ONED) is called by DRIVER to solve two dimensional problem (Eq (29)) (one dimensional problem (Eq (29) with $\partial/\partial z = 0$)). The problem is solved using QN, a non-linear equation solver written by the programmers at Sandia Laboratories. QN solves a series of non-linear algebraic equations of the form

$$F_i(c_1, c_2, \ldots c_i) = 0 \qquad \text{(B1)}$$

for the values of the c_i. The subroutines FOFRZ (FOFR)

called by QN to evaluate these non-linear equations which it does by directly integrating Eq (29) using subroutines RZINTn (RINTn). If QN is able to produce a set of c_i that make all F_i small (less than ERRMAX or 10^{-7} in this case), QN returns to TWOD (ONED) with IFLAG = 2, otherwise, IFLAG is set to some number greater than 2. If, on returning to TWOD (ONED) IFLAG = 2 the electron current ( ) is increased (was less than the limiting current), otherwise, the current is decreased by H (in both cases), the step size. H is then halved and this cycle continues untill H is less than 0.01 (0.001) when control is returned to DRIVER and the next parameter card is processed.

```
      PROGRAM DRIVER(INPUT,OUTPUT)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/COEF/C(201)
      COMMON/RADIUS/R(101)
      COMMON/LENGTH/Z(101)
      REAL K
      NAMELIST/VARIBL/R0,RN,ZN,NR,NZ,V,GAM0,ID
      R0=0.0
      RN=0.0
      ZN=0.0
      GAM0=0.0
    1 CONTINUE
      V=0.0
      ID=0
      READ VARIBL
      IF (EOF(5LINPUT).NE.0) GO TO 999
      IF ((ID.LT.1).OR.(ID.GT.2)) GO TO 900
      IF (RN.LE.R0) GO TO 910
      IF (ID.EQ.1) GO TO 10
      IF ((NR*NZ.EQ.0).OR.(NR*NZ.GT.100)) GO TO 920
   10 CONTINUE
      IF (NR.GT.100) GO TO 930
C
C       GET DATE AND TIME
C
      A=0.0
      B=0.0
      CALL DATE(A)
      CALL TIME(B)
C
C       COMPUTE THE INDEX AT THE INNER BEAM EDGE
C
      NRL1=IFIX(0.9999+R0*FLOAT(NR))+1
      IF (NRL1.LT.2) NRL1=2
      IF (R0.EQ.0.0) NRL1=1
C
C       COMPUTE THE INDEX AT THE OUTER BEAM EDGE
C
      NRL2=IFIX(RN*FLOAT(NR))+1
      IF (NRL2.LE.NRL1) NRL2=NRL1+1
C
C       COMPUTE THE R VALUES
C
      H=1.0/FLOAT(NR)
      NREND=NR+1
      DO 100 I=1,NREND
      R(I)=FLOAT(I-1)*H
  100 CONTINUE
C
C       FORCE THE INNER AND OUTER EDGES TO BE RIGHT
C
      R(NRL1)=R0
      R(NRL2)=RN
      IFLAG=0
      CALL ERXSET(0,0)
      IF (ID.EQ.1) GO TO 200
```

```
C       COMPUTE THE VALUES OF Z AT WHICH WE WANT GAMMA VALUES
C
      NZEND=NZ+1
      K=ZN/(2.0*FLOAT(NZ))
      DO 150 J=1,NZEND
      Z(J)=FLOAT(J-1)*K
  150 CONTINUE
      GO TO 300
C
C       FIND THE ONE DIMENSIONAL LIMITING CURRENT
C
  200 CONTINUE
      CALL ONED(IFLAG)
      C(NR+1)=GAM0
      GO TO 400
C
C       FIND THE TWO DIMENSIONAL CURRENT
C
  300 CONTINUE
      CALL TWOD(IFLAG)
C     GO TO 400
C
C       PRINT THE RESULTS
C
  400 CONTINUE
      PRINT 1000,IFLAG
 1000 FORMAT(1H1,6HIFLAG=,I2)
      PRINT 9000
      IF (ID.EQ.2) GO TO 600
C
C       PRINT THE ONE DIMENSIONAL RESULTS
C
  500 CONTINUE
      PRINT 2000,V,GAM0,R0,RN,A,B
 2000 FORMAT(1H ,"NU=",F10.3,5X,"GAM0=",F7.2,5X,"R0=",F6.4,5X,"RN=",
     1F6.4,5X,"DATE=",A10,5X,"TIME=",A10)
      IF (IFLAG.GT.3) GO TO 1
      PRINT 9000
 9000 FORMAT(1H0)
      DO 550 J=1,NREND
      PRINT 3000,J,R(J),C(J),C(J)-GAM0
 3000 FORMAT(1H "I=",I5,5X,"R=",F6.4,5X,"GAM=",F16.7,5X,"PHI=",F16.7)
  550 CONTINUE
      GO TO 1
C
C       PRINT THE TWO DIMENSIONAL RESULTS
C
  600 CONTINUE
      PRINT 4000,V,GAM0,R0,RN,ZN,A,B
 4000 FORMAT(1H ,"NU=",F9.3,5X,"GAM0=",F7.2,5X,"R0=",F6.4,5X,"RN=",
     1F6.4,5X,"LENGTH=",F9.4,5X,"DATE=",A10,5X,"TIME=",A10)
      IF (IFLAG.GT.3) GO TO 1
      PRINT 9000
      DO 650 J=1,NZEND
      DO 640 I=1,NREND
      CP=GAM0
      IF ((I.NE.NREND).AND.(J.NE.1)) CP=C(J-1+(I-1)*NZ)
```

```
      PRINT 5000,I,J,R(I),Z(J),CP,CP-GAMA
 5000 FORMAT(1H ,"I=",I5,5X,"J=",I5,5X,"R=",F6.4,5X,"Z=",F9.6,5X,"GAM=",
     1F15.7,5X,"PHI=",F15.7)
  640 CONTINUE
      PRINT 9000
  650 CONTINUE
      GO TO 1
  900 CONTINUE
C
C      DIMENSION NOT VALID
C
      PRINT 9001
 9001 FORMAT(1H1,"ID NOT SPECIFIED OR NOT VALID. ID MUST BE EITHER 1 OR
     12. RUN TERMINATED.")
      GO TO 999
C
C      ILLEGAL R0, RN VALUES
C
  910 CONTINUE
      PRINT 9002,R0,RN
 9002 FORMAT(1H1,"STARTING RADIUS MUST BE LESS THAN THE ENDING RADIUS",
     15X,"R0=",F7.4,5X,"RN=",F7.4)
      GO TO 999
C
C      ERROR IN THE NUMBER OF POINTS REQUESTED
C
  920 CONTINUE
      PRINT 9003,NR,NZ
 9003 FORMAT(1H1,"THE PRODUCT OF NR AND NZ MUST BE LESS THAN 100",
     1" AND GREATER THAN 0.  IF MORE POINTS ARE REQUIRED, CHANGES",
     2" MUST BE MADE TO TWOD",/,
     220X,"NR=",I3,5X,"NZ=",I3)
      GO TO 999
C
C      TOO MANY R VALUES REQUESTED
C
  930 CONTINUE
      PRINT 9004,NR
 9004 FORMAT(1H1,"THE NUMBER OF R VALUES REQUESTED MUST BE LESS THAN 100
     1.  IF MORE VALUES ARE REQUIRED CHANGED MUST BE MADE TO ONED",
     25X,"NR=",I4)
  999 CONTINUE
      STOP
      END
```

```
      SUBROUTINE ONED(IFLAG)
      EXTERNAL FOFR
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/COEF/C(1000)
      COMMON/RADIUS/R(101)
      COMMON/QNWORK/DISMAX(100),WORK(21000),IWORK(120)
C
C      CONSTANTS FOR QN
C
      RELERR=1E-7
      ABSERR=0.0
      RES=0.0
      MBAND=1
      VLAST=0.0
      IFLG2=0
      IF (V.EQ.0.0) GO TO 800
      JEND=1
      IFLG2=1
      HV=0.0
      GO TO 850
C
C      FIND THE MAX CURRENT
C
  800 CONTINUE
      HV=GAM0
      DO 855 I=1,NR
      C(I)=GAM0
      DISMAX(I)=GAM0-1.0
  855 CONTINUE
  860 CONTINUE
      VLAST=V
      V=V+HV
      HV=HV*2.0
      CALL QN(FOFR,NR,C,MBAND,DISMAX,RELERR,ABSERR,IFLAG,RES,
     1WORK,IWORK)
      IF (IFLAG.LE.3) GO TO 860
C
C      COMPUTE THE NUMBER OF TIMES THE BINARY SEARCH MUST BE EXICUTED
C      TO GET 0.001 ACCURACY. THEN ADD 1.
C
      JEND=IFIX(ALOG((V-VLAST)/(2.0*0.001))/ALOG(2.0))+1
  850 CONTINUE
      HV=V
      V=2.0*V
      IFLAG=7
C
C      BINARY SEARCH THE CURRENT.  IF QN RETURNS A DID-NOT-CONVERGE
C      RETURN CODE ASSUME THE CURRENT IS TOO HIGH. ELSE THE CURRENT IS
C      TOO LOW
C
      DO 200 J=1,JEND
      DO 300 I=1,NR
      DISMAX(I)=GAM0-1.0
  300 CONTINUE
      IF(IFLAG.LE.3)  GO TO 500
      DO 100 I=1,NR
      C(I)=GAM0
```

```
  100 CONTINUE
  500 CONTINUE
C
C       FOR THE PURPOSE OF DOING THE BINARY SEARCH A REDUCTION OF
C       10 ORDERS OF MAGNITUDE IS CLOSE ENOUGH
C
      IF(IFLAG.LE.3) V=V+HV
      IF(IFLAG.GT.3) V=V-HV
      HV=HV/2.0
      IFLAG=1
  400 CONTINUE
      CALL QN(FOFR,NR,C,MBAND,DISMAX,RELERR,ABSERR,IFLAG,RES,
     1WORK,IWORK)
      IF(IFLAG.LE.3) VLAST=V
  200 CONTINUE
C
C       FOR THE FINAL PASS THE ERROR MUST BE AS SMALL AS POSSIBLE
C
      IF(IFLAG.LT.3) GO TO 900
      IF(IFLAG.EQ.3) GO TO 700
      IF (IFLG2.EQ.1) GO TO 900
      V=VLAST
      DO 600 I=1,NR
      C(I)=GAM0
      DISMAX(I)=GAM0-1.0
  600 CONTINUE
      IFLAG=1
  700 CONTINUE
      CALL QN(FOFR,NR,C,MBAND,DISMAX,RELERR,ABSERR,IFLAG,RES,
     1WORK,IWORK)
      IF (IFLAG.EQ.3) GO TO 700
  900 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE FOFR(C,F)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/RADIUS/R(101)
      COMMON/SUB/IFLAG,M,G(7)
      DIMENSION F(5000),C(5000)
      M=7
      DO 800 I=1,NR
C
C      INSURE THAT ALL C(I) ARE GREATER THAN 1.0
C
      IF(C(I).LE.1.0) GO TO 900
C
C      INTEGRATE THE SECOND HALF INTERVAL
C
      F(I)=RINT2(I,C)
      IF (IFLAG.NE.0) GO TO 900
C
C      IF R IS GREATER THAN 0.0 INTEGRATE THE FIRST HALF OF THE
C      INTERVAL
C
      IF (I.GT.1) F(I)=F(I)+RINT1(I,C)
      IF (IFLAG.NE.0) GO TO 900
  800 CONTINUE
      GO TO 999
C
C      ASSUME THAT V IS GREATER THAN V(LIMIT)
C      AND FORCE QN TO CALL THE SERIES DIVERGING
C
  900 CONTINUE
      DO 901 I=1,NR
      F(I)=1.0
  901 CONTINUE
  999 CONTINUE
      RETURN
      END
```

```
      FUNCTION RINT1(I,C)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/RADIUS/R(101)
      COMMON/SUB/IFLAG,M,G(7)
      DIMENSION C(201)
      H=R(I)-R(I-1)
C
C      THE DERIVITIVES ARE CONSTANT OVER EACH SUB INTERVAL
C
      DRI=1.0/(R(I)-R(I-1))
      DRI1=-DRI
C
C      COMPUTE THE FUNCTION VALUES AT EACH POINT IN THE SUB INTERVAL
C      FOR THE SIMPSONS RULE INTEGRAL
C
      DO 100 J=1,M
      RA=R(I-1)+(FLOAT(J-1)*H)/(FLOAT(M-1))
      G(J)=(C(I-1)*DRI1+C(I)*DRI)*DRI*RA
C
C      IF WE ARE IN THE PART OF THE TUBE OCCUPIED BY THE BEAM
C      ADD IN THE NON-LINEAR TERM
C
      IF ((I.LE.NRL1).OR.(I.GT.NRL2)) GO TO 100
      RI=(RA-R(I-1))/(R(I)-R(I-1))
      RI1=(R(I)-RA)/(R(I)-R(I-1))
      GAM=C(I-1)*RI1+C(I)*RI
      IF (GAM.LE.1.0) GO TO 900
      G(J)=G(J)+(4.0*V*GAM*RI*RA)/((SQRT(GAM**2-1.0))*(RN**2-R0**2))
  100 CONTINUE
C
C      DO A SIMPSON RULE INTEGRAL
C
      RINT1=G(1)
      MEND=M-1
      DO 200 J=2,MEND,2
      RINT1=RINT1+4.0*G(J)+2.0*G(J+1)
  200 CONTINUE
      RINT1=RINT1-G(M)
      RINT1=RINT1*(H/(3.0*FLOAT(M-1)))
      IFLAG=0
      GO TO 999
C
C     NEGATIVE SQUARE ROOT///NOTIFY CALLING PROGRAM
C
  900 CONTINUE
      RINT1=0.0
      IFLAG=1
  999 CONTINUE
      RETURN
      END
```

```
      FUNCTION RINT2(I,C)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/RADIUS/R(101)
      COMMON/SUB/IFLAG,M,G(7)
      DIMENSION C(201)
      H=R(I+1)-R(I)
C
C      THE DERIVITIVES ARE CONSTANT OVER EACH SUB INTERVAL
C
      DRI=-1.0/(R(I+1)-R(I))
      DRI1=-DRI
C
C      COMPUTE THE FUNCTION VALUES AT EACH POINT IN THE SUB INTERVAL
C      FOR THE SIMPSONS RULE INTEGRAL
C
      DO 100 J=1,M
      RA=R(I)+(FLOAT(J-1)*H)/(FLOAT(M-1))
      G(J)=(C(I)*DRI+GAM0*DRI1)*DRI*RA
      IF (I.LT.NR) G(J)=(C(I)*DRI+C(I+1)*DRI1)*DRI*RA
C
C      IF WE ARE IN THE PART OF THE TUBE OCCUPIED BY THE BEAM
C      ADD IN THE NON-LINEAR TERM
C
      IF ((I.LT.NRL1).OR.(I.GE.NRL2)) GO TO 100
      RI=(R(I+1)-RA)/(R(I+1)-R(I))
      RI1=(RA-R(I))/(R(I+1)-R(I))
      GAM=C(I)*RI+GAM0*RI1
      IF (I.LT.NR) GAM=C(I)*RI+C(I+1)*RI1
      IF (GAM.LE.1.0) GO TO 900
      G(J)=G(J)+(4.0*V*GAM*RI*RA)/((SQRT(GAM**2-1.0))*(RN**2-R0**2))
  100 CONTINUE
C
C      DO A SIMPSON RULE INTEGRAL
C
      RINT2=G(1)
      MEND=M-1
      DO 200 J=2,MEND,2
      RINT2=RINT2+4.0*G(J)+2.0*G(J+1)
  200 CONTINUE
      RINT2=RINT2-G(M)
      RINT2=RINT2*(H/(3.0*FLOAT(M-1)))
      IFLAG=0
      GO TO 999
C
C     NEGATIVE SQUARE ROOT///NOTIFY CALLING PROGRAM
C
  900 CONTINUE
      RINT2=0.0
      IFLAG=1
  999 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE TWOD(IFLAG)
      EXTERNAL FOFRZ
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/RADIUS/R(11)
      COMMON/LENGTH/Z(11)
      COMMON/COEF/C(101)
      COMMON/QNWORK/DISMAX(100),WORK(21000),IWORK(120)
      IFLG2=0
      IF (V.EQ.0.0) IFLG2=1
      PRINT 9999
 9999 FORMAT(1H1)
      VLAST=V
      HV=GAM0
C
C       CONSTANTS FOR QN
C
      MBAND=NZ+1
      RELERR=1E-7
      ABSERR=0.0
      RES=0.0
C
C       CHECK FOR ONE PASS PNLY
C
      IFLAG=7
      IF (IFLG2.EQ.0) GO TO 600
C
C       COMPUTE THE INITIAL TWO DIMENSIONAL GUESS
C
   50 CONTINUE
      NEND=NR*NZ
      DO 100 I=1,NEND
      C(I)=GAM0
  100 CONTINUE
C
C       FIND THE MAX CURRENT
C
  150 CONTINUE
      DO 200 N=1,NEND
      DISMAX(N)=GAM0-1.0
  200 CONTINUE
      IFLAG=1
      VLAST=V
      V=V+HV
      HV=HV*2.0
      CALL QN(FOFRZ,NR*NZ,C,MBAND,DISMAX,RELERR,ABSERR,IFLAG,RES,
     1WORK,IWORK)
      IF (IFLAG.LE.3) GO TO 150
      IF (IFLAG.EQ.9) GO TO 999
C
C       FOUND THE MAX CURRENT
C

  300 CONTINUE
      HV=(V-VLAST)
      NMAX=IFIX(ALOG((V-VLAST)/(0.01))/ALOG(2.0))+1
```

```
C
C      DO A BINARY SEARCH FOR THE LIMITING CURRENT
C              IF QN RETURNS A DID-NOT-CONVERGE RETURN CODE V IS
C              ASSUMED TO BE LARGER THAN THE LIMITING CURRENT
C
      DO 500 N=1,NMAX
      DO 350 I=1,NEND
      DISMAX(I)=GAM0-1.0
  350 CONTINUE
      IF (IFLAG.LE.3) GO TO 400
      DO 380 I=1,NEND
      C(I)=GAM0
  380 CONTINUE
  400 CONTINUE
C
C      FOR THE BINARY SEARCH ROUTINE A REDUCTION OF 10 ORDERS OF
C      MAGNITUDE IS CLOSE ENOUGH
C
      HV=HV/2.0
      IF (IFLAG.LE.3) VLAST=V
      IF (IFLAG.LE.3) V=V+HV
      IF (IFLAG.GT.3) V=V-HV
      IFLAG=1
      CALL QN(FOFRZ,NR*NZ,C,MBAND,DISMAX,RELERR,ABSERR,IFLAG,RES,
     1WORK,IWORK)
  500 CONTINUE
C
C      INSURE THAT THE LAST GOOD CURRENT IS RETURNED TO THE CALLER
C      AND RETURN
C
  600 CONTINUE
      IF (IFLAG.LE.2) GO TO 999
      IF(IFLAG.EQ.3) GO TO 700
      V=VLAST
      NEND=NR*NZ
      DO 650 I=1,NEND
      DISMAX(I)=GAM0-1.0
      C(I)=GAM0
  650 CONTINUE
      IFLAG=1
  700 CONTINUE
      CALL QN(FOFRZ,NR*NZ,C,MBAND,DISMAX,RELERR,ABSERR,IFLAG,RES,
     1WORK,IWORK)
      IF (IFLAG.EQ.3) GO TO 700
  999 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE FOFRZ(C,F)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/RADIUS/R(101)
      COMMON/LENGTH/Z(101)
      COMMON/SUB2/IFLAG,M,H,K,G(7,7)
      REAL K
      DIMENSION C(1000),F(1000)
      M=7
      DO 200 I=1,NR
      DO 100 J=1,NZ
C
C      CHECK TO INSURE THAT EACH VALUE OF GAMMA IS GREATER THAN 1.0
C
      IF (C(J+(I-1)*NZ).LE.1.0) GO TO 900
C
C     INETGRATE OVER QUADRANT 2
C
      F(J+(I-1)*NZ)=RZINT2(I,J,C)
      IF (IFLAG.NE.0) GO TO 900
C
C      INTEGRATE OVER QUADRANT 1 UNLESS Z.GT.L/2
C
      IF (J.LT.NZ) F(J+(I-1)*NZ)=F(J+(I-1)*NZ)+RZINT1(I,J,C)
      IF (IFLAG.NE.0) GO TO 900
C
C      INTEGRATE OVER QUADRANT 3 UNLESS R IS TOO SMALL
C
      IF (I.GT.1) F(J+(I-1)*NZ)=F(J+(I-1)*NZ)+RZINT3(I,J,C)
      IF (IFLAG.NE.0) GO TO 900
C
C      INTEGRATE OVER QUADRANT 4 UNLESS .....
C
      IF ((I.GT.1).AND.(J.LT.NZ))
     1         F(J+(I-1)*NZ)=F(J+(I-1)*NZ)+RZINT4(I,J,C)
      IF (IFLAG.NE.0) GO TO 900
  100 CONTINUE
  200 CONTINUE
      RETURN
C
C      NEGATIVE SQUARE ROOT////ASSUME V IS GREATER THAN V LIMIT
C
  900 CONTINUE
      IEND=NR*NZ
      DO 910 I=1,IEND
      F(I)=1.0
  910 CONTINUE
      RETURN
      END
```

```
      REAL FUNCTION RZINT1(I,J,C)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/RADIUS/R(101)
      COMMON/LENGTH/Z(101)
      COMMON/SUB2/IFLAG,M,H,K,G(7,7)
      DIMENSION C(1000)
      REAL K
C
C     QUADRANT 1 INTEGRATOR
C
      RZINT1=0.0
      H=(R(I+1)-R(I))/FLOAT(M-1)
      K=(Z(J+2)-Z(J+1))/FLOAT(M-1)
      DRI=-1.0/(R(I+1)-R(I))
      DRI1=-DRI
      DZJ=-1.0/(Z(J+2)-Z(J+1))
      DZJ1=-DZJ
      DO 200 L=1,M
      RA=R(I)+FLOAT(L-1)*H
      RI=(R(I+1)-RA)/(R(I+1)-R(I))
      RI1=(RA-R(I))/(R(I+1)-R(I))
      DO 100 N=1,M
      ZA=Z(J+1)+FLOAT(N-1)*K
      ZJ=(Z(J+2)-ZA)/(Z(J+2)-Z(J+1))
      ZJ1=(ZA-Z(J+1))/(Z(J+2)-Z(J+1))
      C1=C(J+(I-1)*NZ)
      C2=C(J+1+(I-1)*NZ)
      C3=GAM0
      IF (I.LT.NR) C3=C(J+I*NZ)
      C4=GAM0
      IF (I.LT.NR) C4=C(J+1+I*NZ)
      F1=(ZJ**2)*(DRI**2)+(RI**2)*(DZJ**2)
      F2=(ZJ*ZJ1)*(DRI**2)+(RI**2)*(DZJ*DZJ1)
      F3=(ZJ**2)*(DRI*DRI1)+(RI*RI1)*(DZJ**2)
      F4=(ZJ1*ZJ)*(DRI*DRI1)+(RI1*RI)*(DZJ1*DZJ)
      G(L,N)=(C1*F1+C2*F2+C3*F3+C4*F4)*RA
      IF ((I.LT.NRL1).OR.(I.GE.NRL2)) GO TO 100
      GAM=C1*RI*ZJ+C2*RI*ZJ1+C3*RI1*ZJ+C4*RI1*ZJ1
      IF (GAM.LE.1.0) GO TO 900
      RADIC=(4.0*V*RA*GAM)/((SQRT(GAM**2-1.0))*(RN**2-R0**2))
      G(L,N)=G(L,N)+RADIC*RI*ZJ
  100 CONTINUE
  200 CONTINUE
C
C      DO A TRAPAZOIDAL RULE INTEGRAL
C
      CALL INTGL(RZINT1)
      IFLAG=0
      RETURN
C
C      NEGATIVE OR ZERO SQUARE ROOT/////NOTIFY CALLING PROGRAM
C
  900 CONTINUE
      IFLAG=1
      RETURN
      END
```

```
      REAL FUNCTION RZINT2(I,J,C)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/LENGTH/Z(101)
      COMMON/RADIUS/R(101)
      COMMON/SUB2/IFLAG,M,H,K,G(7,7)
      REAL K
C
C      QUADRANT 2 INTEGRATOR
C
      DIMENSION C(1000)
      RZINT2=0.0
      H=(R(I+1)-R(I))/FLOAT(M-1)
      K=(Z(J+1)-Z(J))/FLOAT(M-1)
      DRI=-1.0/(R(I+1)-R(I))
      DRI1=-DRI
      DZJ=1.0/(Z(J+1)-Z(J))
      DZJ1=-DZJ
      DO 200 L=1,M
      RA=R(I)+FLOAT(L-1)*H
      RI=(R(I+1)-RA)/(R(I+1)-R(I))
      RI1=(RA-R(I))/(R(I+1)-R(I))
      DO 100 N=1,M
      ZA=Z(J)+FLOAT(N-1)*K
      ZJ=(ZA-Z(J))/(Z(J+1)-Z(J))
      ZJ1=(Z(J+1)-ZA)/(Z(J+1)-Z(J))
      C1=GAM0
      IF (J.GT.1) C1=C(J-1+(I-1)*NZ)
      C2=C(J+(I-1)*NZ)
      C3=GAM0
      IF ((J.GT.1).AND.(I.LT.NR)) C3=C(J-1+I*NZ)
      C4=GAM0
      IF (I.LT.NR) C4=C(J+I*NZ)
      F1=(ZJ1*ZJ)*(DRI**2)+(RI**2)*(DZJ1*DZJ)
      F2=(ZJ**2)*(DRI**2)+(RI**2)*(DZJ**2)
      F3=(ZJ1*ZJ)*(DRI1*DRI)+(RI1*RI)*(DZJ1*DZJ)
      F4=(ZJ**2)*(DRI1*DRI)+(RI1*RI)*(DZJ**2)
      G(L,N)=(C1*F1+C2*F2+C3*F3+C4*F4)*RA
      IF ((I.LT.NRL1).OR.(I.GE.NRL2)) GO TO 100
      GAM=C1*RI*ZJ1+C2*RI*ZJ+C3*RI1*ZJ1+C4*RI1*ZJ
      IF (GAM.LE.1.0) GO TO 900
      RADIC=(4.0*V*RA*GAM)/((SQRT(GAM**2-1.0))*(RN**2-R0**2))
      G(L,N)=G(L,N)+RADIC*RI*ZJ
  100 CONTINUE
  200 CONTINUE
C
C      DO A TRAPAZOIDAL RULE INTEGRAL
C
      CALL INTGL(RZINT2)
      IFLAG=0
      RETURN
C
C      NEGATIVE OR ZERO SQUARE ROOT/////NOTIFY CALLING PROGRAM
C
  900 CONTINUE
      IFLAG=1
      RETURN
      END
```

```
      REAL FUNCTION RZINT3(I,J,C)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/LENGTH/Z(101)
      COMMON/RADIUS/R(101)
      COMMON/SUB2/IFLAG,M,H,K,G(7,7)
      REAL K
C
C     QUADRANT 3 INTEGRATOR
C
      DIMENSION C(1000)
      RZINT3=0.0
      H=(R(I)-R(I-1))/FLOAT(M-1)
      K=(Z(J+1)-Z(J))/FLOAT(M-1)
      DRI=1.0/(R(I)-R(I-1))
      DRI1=-DRI
      DZJ=1.0/(Z(J+1)-Z(J))
      DZJ1=-DZJ
      DO 200 L=1,M
      RA=R(I-1)+FLOAT(L-1)*H
      RI=(RA-R(I-1))/(R(I)-R(I-1))
      RI1=(R(I)-RA)/(R(I)-R(I-1))
      DO 100 N=1,M
      ZA=Z(J)+FLOAT(N-1)*K
      ZJ=(ZA-Z(J))/(Z(J+1)-Z(J))
      ZJ1=(Z(J+1)-ZA)/(Z(J+1)-Z(J))
      C1=GAM0
      IF (J.GT.1) C1=C(J-1+(I-2)*NZ)
      C2=C(J+(I-2)*NZ)
      C3=GAM0
      IF (J.GT.1) C3=C(J-1+(I-1)*NZ)
      C4=C(J+(I-1)*NZ)
      F1=(ZJ1*ZJ)*(DRI1*DRI)+(RI1*RI)*(DZJ1*DZJ)
      F2=(ZJ**2)*(DRI1*DRI)+(RI1*RI)*(DZJ**2)
      F3=(ZJ1*ZJ)*(DRI**2)+(RI**2)*(DZJ1*DZJ)
      F4=(ZJ**2)*(DRI**2)+(RI**2)*(DZJ**2)
      G(L,N)=(C1*F1+C2*F2+C3*F3+C4*F4)*RA
      IF ((I.LE.NRL1).OR.(I.GT.NRL2)) GO TO 100
      GAM=C1*RI1*ZJ1+C2*RI1*ZJ+C3*RI*ZJ1+C4*RI*ZJ
      IF (GAM.LE.1.0) GO TO 900
      RADIC=(4.0*V*RA*GAM)/((SQRT(GAM**2-1.0))*(RN**2-R0**2))
      G(L,N)=G(L,N)+RADIC*RI*ZJ
  100 CONTINUE
  200 CONTINUE
C
C      DO A TRAPAZOIDAL RULE INTEGRAL
C
      CALL INTGL(RZINT3)
      IFLAG=0
      RETURN
C
C      NEGATIVE OR ZERO SQUARE ROOT/////NOTIFY CALLING PROGRAM
C
  900 CONTINUE
      IFLAG=1
      RETURN
      END
```

```
      REAL FUNCTION RZINT4(I,J,C)
      COMMON/DATA/R0,RN,ZN,NR,NRL1,NRL2,NZ,V,GAM0
      COMMON/RADIUS/R(101)
      COMMON/LENGTH/Z(101)
      COMMON/SUB2/IFLAG,M,H,K,G(7,7)
      REAL K
C
C     QUADRANT 4 INTEGRATOR
C
      DIMENSION C(1000)
      RZINT3=4.0
      H=(R(I)-R(I-1))/FLOAT(M-1)
      K=(Z(J+2)-Z(J+1))/FLOAT(M-1)
      DRI=1.0/(R(I)-R(I-1))
      DRI1=-DRI
      DZJ=-1.0/(Z(J+2)-Z(J+1))
      DZJ1=-DZJ
      DO 200 L=1,M
      RA=R(I-1)+FLOAT(L-1)*H
      RI=(RA-R(I-1))/(R(I)-R(I-1))
      RI1=(R(I)-RA)/(R(I)-R(I-1))
      DO 100 N=1,M
      ZA=Z(J+1)+FLOAT(N-1)*K
      ZJ=(Z(J+2)-ZA)/(Z(J+2)-Z(J+1))
      ZJ1=(ZA-Z(J+1))/(Z(J+2)-Z(J+1))
      C1=C(J+(I-2)*NZ)
      C2=C(J+1+(I-2)*NZ)
      C3=C(J+(I-1)*NZ)
      C4=C(J+1+(I-1)*NZ)
      F1=(ZJ**2)*(DRI1*DRI)+(RI1*RI)*(DZJ**2)
      F2=(ZJ1*ZJ)*(DRI1*DRI)+(RI1*RI)*(DZJ1*DZJ)
      F3=(ZJ**2)*(DRI**2)+(RI**2)*(DZJ**2)
      F4=(ZJ1*ZJ)*(DRI**2)+(RI**2)*(DZJ1*DZJ)
      G(L,N)=(C1*F1+C2*F2+C3*F3+C4*F4)*RA
      IF ((I.LE.NRL1).OR.(I.GT.NRL2)) GO TO 100
      GAM=C1*RI1*ZJ+C2*RI1*ZJ1+C3*RI*ZJ+C4*RI*ZJ1
      IF (GAM.LE.1.0) GO TO 900
      RADIC=(4.0*V*RA*GAM)/((SQRT(GAM**2-1.0))*(RN**2-R0**2))
      G(L,N)=G(L,N)+RADIC*RI*ZJ
  100 CONTINUE
  200 CONTINUE
C
C      DO A TRAPAZOIDAL RULE INTEGRAL
C
      CALL INTGL(RZINT4)
      IFLAG=0
      RETURN
C
C      NEGATIVE OR ZERO SQUARE ROOT/////NOTIFY CALLING PROGRAM
C
  900 CONTINUE
      IFLAG=1
      RETURN
      END
```

```
      SUBROUTINE INTGL(RESULT)
C
C      DO A TRAPAZOIDAL INTEGRAL APPROXIMATION ON THE DATA IN G
C
      COMMON/SUB2/IFLAG,M,H,K,G(7,7)
      REAL K
      RESULT=(1.0/4.0)*(G(1,1)+G(1,M)+G(M,1)+G(M,M))
      JEND=M-1
      DO 100 J=2,JEND
      RESULT=RESULT+(1.0/2.0)*(G(1,J)+G(J,1)+G(M,J)+G(J,M))
  100 CONTINUE
      IEND=JEND
      DO 300 I=2,IEND
      DO 200 J=2,JEND
      RESULT=RESULT+G(I,J)
  200 CONTINUE
  300 CONTINUE
      RESULT=H*K*RESULT
      RETURN
      END
```

Vita

Steven M. Harrington was born on 19 August 1948 in Seattle Washington. He graduated from high school in 1966 and entered the United States Air Force. He attended Auburn University under the AECP program and recieved the degree of Bachelor of Physics in December 1973. Upon graduation he entered OTS and was commissioned in April 1974 He entered the School of Engineering, Air Force Institute of Technology in June 1977.

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The equation governing the steady state potential distribution created by a cold unneutralized annular relativistic electron beam propagating axially within a finite length drift tube with grounded anode foil, collector plate, and walls and immersed in an axially directed infinite magnetic guide field is solved numerically using the method of finite elements. These numerical results are used to check the accuracy of available analytical approximations to the limiting current.

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